

PERTURBATIONS OF QUADRATIC CENTERS

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ABSTRACT. – We study the bifurcation of limit cycles in general quadratic perturbations of plane quadratic vector fields having a center at the origin. For any of the cases, we determine the essential perturbation and compute the corresponding bifurcation function. As an application, we find the precise location of the subset of centers in Q_3^R surrounded by period annuli of cyclicity at least three. Two specific cases are considered in more detail: the isochronous center S_1 and one of the intersection points (Q_4^+) of Q_4 and Q_3^R . We prove that the period annuli around S_1 and Q_4^+ have cyclicity two and three respectively. The proof is based on the possibility to derive appropriate Picard-Fuchs equations satisfied by the independent integrals included in the related bifurcation function. © Elsevier, Paris

1. Introduction

This paper is concerned with the bifurcation of limit cycles in plane quadratic systems under small quadratic perturbations. We assume that the unperturbed system has at least one center. It is well known that each quadratic system with a center also possesses an explicit first integral

(*) Manuscript presented by J.-P. FRANÇOISE, received in February 1996.

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Supported by Bulgarian NFSR Grant MM-428/94.

(constant of motion) defined and analytic at least in an open domain around the center. This is the reason why such systems are called integrable. Each center is surrounded by a continuous set of periodic trajectories (the period annulus). In quadratic systems, period annuli begin at a center and terminate at a separatrix contour (on the Poincaré sphere) containing at least one singular point. They are up to two centers in a quadratic integrable system. The conditions for a quadratic system to have a center are known since the beginning of the century [12]. The quadratic centers are divided into several types. The most simple classification can be obtained by using complex variables [21], [28]. Taking a complex coordinate $z = x + iy$, one can express any plane quadratic system having a weak focus at the origin in the form

$$\dot{z} = -iz + Az^2 + B|z|^2 + C\bar{z}^2, \quad A, B, C \in \mathbb{C}.$$

Moreover, if $B \neq 0$ we can suppose that $B = 2$ (performing a suitable rotation and scaling of coordinates). Similarly, if $B = 0$ but $A \neq 0$ we take $A = 1$ and when $A = B = 0$ we take $C = 1$. The last two systems are integrable (that is $\dot{z} = 0$ is a center). In the case where $B = 2$, the origin is a center if and only if one of the following conditions holds: (i) $A = -1$, (ii) A and C are real, (iii) $A = 4$, $|C| = 2$. Using the terminology from [28], the list of quadratic centers at $z = 0$ looks hence as follows:

$$\begin{aligned} \dot{z} &= -iz - z^2 + 2|z|^2 + (b + ic)\bar{z}^2, & \text{Hamiltonian } (Q_3^H) \\ \dot{z} &= -iz + az^2 + 2|z|^2 + b\bar{z}^2, & \text{reversible } (Q_3^R) \\ \dot{z} &= -iz + 4z^2 + 2|z|^2 + (b + ic)\bar{z}^2, |b + ic| = 2, & \text{codimension four } (Q_4) \\ \dot{z} &= -iz + z^2 + (b + ic)\bar{z}^2, & \text{generalized Lotka-Volterra } (Q_3^{LV}) \\ \dot{z} &= -iz + \bar{z}^2, & \text{Hamiltonian triangle} \end{aligned}$$

In these equations a, b, c stand for arbitrary real constants. The geometric description of the reversible centers is quite simple: all they have a first integral of the form $H(x^2, y) = \text{const}$. That is, their phase portraits possess axial symmetry. In fact, the centers from Q_3^H , Q_4 , Q_3^{LV} with $c = 0$ and the Hamiltonian triangle are reversible as well. Because of the symmetry possessed in addition, we shall refer to them as the *degenerate cases*. As mentioned in [28], all the degenerate cases need a special treatment. Apart of these, we will consider all the remaining centers as generic ones.

Let

$$(1) \quad \begin{cases} \dot{x} = H_y/M, \\ \dot{y} = -H_x/M \end{cases}$$

be any of the above systems rewritten in (x, y) coordinates. Here H is the constant of motion and M is the integrating factor ($M = 1$ in the Hamiltonian case). Consider a small quadratic perturbation of (1):

$$(2) \quad \begin{cases} \dot{x} = H_y/M + \varepsilon f(x, y, \varepsilon), \\ \dot{y} = -H_x/M + \varepsilon g(x, y, \varepsilon). \end{cases}$$

In this system ε is a small parameter and f, g are quadratic polynomials in x, y with coefficients depending analytically on ε . The question we consider is how many limit cycles can be born out for small ε from the period annulus surrounding the center at the origin. In his celebrated paper, BAUTIN [1] found that at most three limit cycles can appear near a focus or a center of any quadratic system. Apart of that, the question about the number of limit cycles appearing near separatrix contours or bifurcating out of a period annulus of a quadratic integrable system is still generally unsolved. In this direction, the most general results concerning the limit cycles in (2) were obtained very recently by ŻOŁĄDEK [28] for perturbations of Q_3^{LV} , if there are two or three invariant lines in the unperturbed system (1), and by HOROZOV and the author [7] in the Hamiltonian case Q_3^H , provided the unperturbed system has three saddles and one center. It is easy to see by inspecting the curves Γ_2, Γ_∞ in the parameter space for which (1) has a double critical point (resp. a critical point escaped to infinity) that the result in [28] holds for parameter values (b, c) in Q_3^{LV} satisfying $(b, c) \neq (-1, 0), (\frac{1}{3}, 0)$ and $(b^2 + c^2 - \frac{1}{3})^2 + \frac{4}{27}(2b - 1) \geq 0$. A similar observation shows that the result from [7] applies to all Hamiltonian systems Q_3^H with parameters (b, c) taken from the set $(b^2 + c^2 - 3)^2 - 8b - 12 > 0, c \neq 0$. See the bifurcation diagram of Q_3^H, Q_3^R and Q_3^{LV} in Figures 1, 2, 3. In [7] and [28] it is supposed that the perturbation functions f, g do not depend on ε . However, it can be easily seen that the results cited are immediately applicable to the general quadratic perturbation as taken in (2), see below. Somewhat less general results have been obtained [8] for the symmetric Hamiltonian case with both two saddles and centers (analytically, $b^2 + c^2 = 1, c \neq 0$),

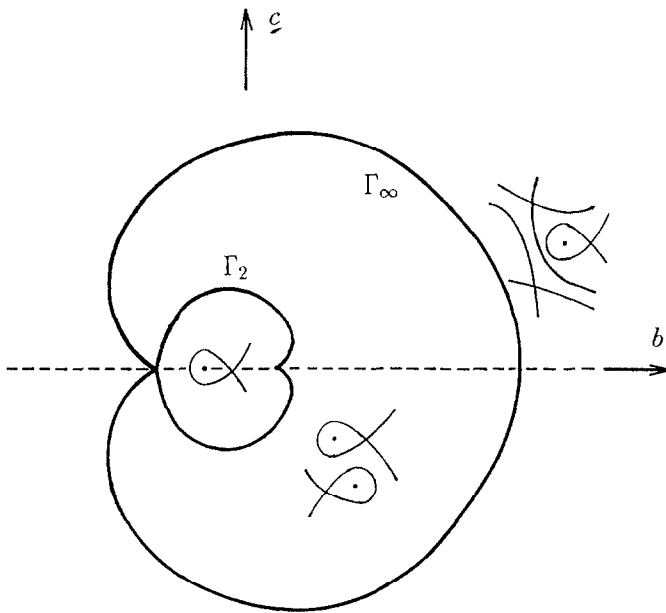


Fig. 1. – Bifurcation diagram of Q_3^H in the (b, c) -plane. The curves Γ_2 and Γ_∞ have equations $\Gamma_2 : (b^2 + c^2 - \frac{1}{3})^2 + \frac{4}{27}(2b - 1) = 0$, $\Gamma_\infty : (b^2 + c^2 - 3)^2 - 8b - 12 = 0$. The dotted line $c = 0$ corresponds to the degenerate cases $Q_3^H \cap Q_3^R$.

and for the Hamiltonian case with one center and two saddles [27] (the latter is determined by the restriction $(b^2 + c^2 - 3)^2 - 8b - 12 = 0$, $c \neq 0$). In all these cases, the exact upper bound for the number of limit cycles emerging from the period annulus for ε sufficiently small is proven to be two. In a recent paper [10] it was proved that the cyclicity of the period annulus of the Hamiltonian triangle is equal to three. As usual, we use the notion of *cyclicity* for the total number of limit cycles which can emerge from a configuration of trajectories (center point, period annulus, separatrix cycle) under a perturbation.

Another results concerned with certain specific quadratic centers can be found in [3], [6], [9], [22], [23]. In particular, CHICONE and JACOBS [3] investigated the isochronous centers $S_1 \in Q_3^{LV}$ and $S_k \in Q_3^R$, $k = 2, 3, 4$. They found that the cyclicity of the annulus is 1 for S_1 and 2 for the other isochrones. However, the result concerning S_1 is not correct. Below we prove that the right bound in this case is also two.

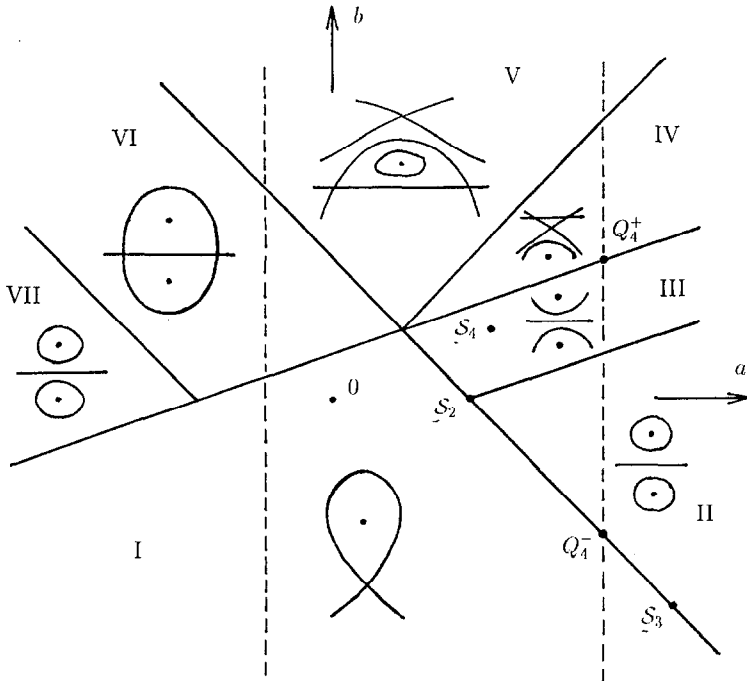


Fig. 2. – Bifurcation diagram of Q_3^R in the (a, b) -plane. The equations of lines in the diagram are $a - 3b \pm 2 = 0$, $a + b \pm 2 = 0$, $a = b$. The dotted line $a = 4$ corresponds to cyclicity three centers. The dotted line $a = -1$ corresponds to the intersection with Q_3^H . The points Q_4^\pm are the intersection with the codimension four case Q_4 . Points S_j stand for the corresponding isochronous centers.

In general, nothing is known up to now about the system Q_4 and almost nothing about Q_3^R . The expected maximal cyclicity of annuli for these cases is three [28], see in this connection the conjecture below.

A key point in the investigation the number of perturbing limit cycles in (2) is to reduce the original problem to the generally considered to be less difficult problem about counting the zeros of a suitable perturbation function. At first order (in ε), this is the divergence integral (also known as the Melnikov function)

$$\begin{aligned} (3) \quad M_1(h) &= - \int \int_{H < h} [(Mf)_x + (Mg)_y] dx dy|_{\varepsilon=0} \\ &= \int_{H=h} M [g(x, y, 0) dx - f(x, y, 0) dy] \end{aligned}$$

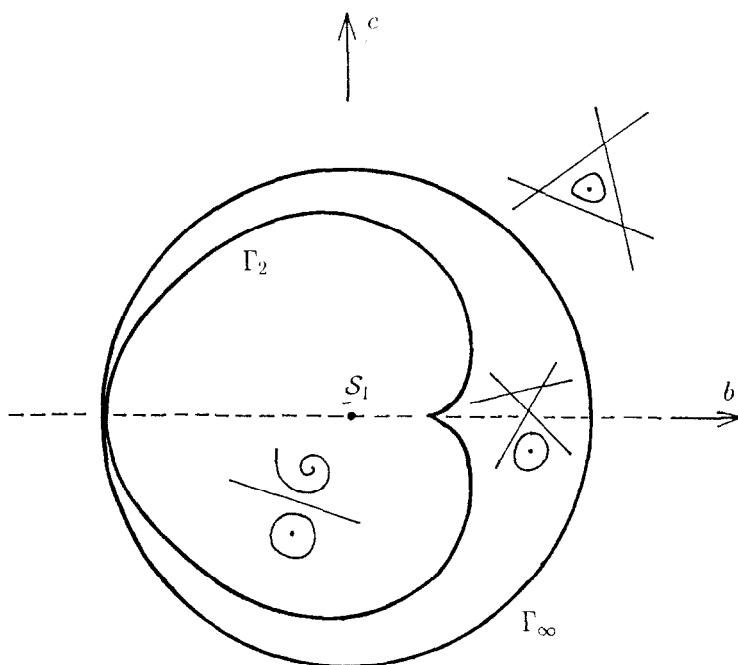


Fig. 3. – Bifurcation diagram of Q_3^{LV} in the (b, c) -plane. The curves Γ_2 and Γ_∞ have equations $\Gamma_2 : (b^2 + c^2 - \frac{1}{3})^2 + \frac{4}{27}(2b - 1) = 0$, $\Gamma_\infty : b^2 + c^2 - 1 = 0$. The dotted line $c = 0$ corresponds to the degenerate cases. The origin corresponds to the isochronous center S_1 .

considered for $h \in (h_c, h_s)$. Here h_c is the critical level of H corresponding to the center $z = 0$ and h_s denotes the value of H for which the period annulus terminates at the separatrix polycycle. We can ensure that $h_s > h_c$ choosing M to be positive. Recall that the Melnikov integral appears as the coefficient at ε in the expansion of the displacement function:

$$(4) \quad d(h, \varepsilon) = \varepsilon M_1(h) + O(\varepsilon^2).$$

As an example, let us take the Hamiltonian case. The integral $M_1(h)$ becomes

$$M_1(h) = - \int \int_{H < h} [\alpha x + \beta y + \gamma] dx dy = \int_{H=h} \left[\alpha xy + \frac{1}{2} \beta y^2 + \gamma y \right] dx$$

where $\alpha x + \beta y + \gamma = f_x(x, y, 0) + g_y(x, y, 0)$. Therefore, it suffices to consider only the particular perturbation

$$\begin{aligned}\dot{x} &= H_y, \\ \dot{y} &= -H_x + \varepsilon \left(\alpha xy + \frac{1}{2} \beta y^2 + \gamma y \right)\end{aligned}$$

in order to study (at first order in ε) the possible number of limit cycles for the entire class of perturbations having the same divergence.

If one wants to consider bifurcations of any order in ε however, then at least two problems appear. The first one is caused by the fact that in the degenerate cases the Melnikov function $M_1(h)$ can *never* yield the maximum number of zeros possessed by $d(h, \varepsilon)$ for the whole class of perturbations even if f, g do not depend on ε . The reason is that the degenerate cases admit additional symmetries (being reversible too) which results in a lower bound for the number of zeros possessed by $M_1(h)$ than the expected one. As an example we can take the Hamiltonian triangle, in which case $M_1(h)$ has no isolated zeros at all. Therefore, for centers like that, we have necessarily to consider in more detail the class of perturbations with $M_1(h) \equiv 0$ and then to try to evaluate the next term $M_2(h)$ in the expansion of $d(h, \varepsilon)$:

$$(5) \quad d(h, \varepsilon) = \varepsilon^2 M_2(h) + O(\varepsilon^3),$$

and so on. However, the computation of M_k , $k \geq 2$ is often a challenging technical exercise. Algorithms to several cases are given in [5], [9], [24], [26], [28].

The second problem is what happens in the generic cases when $M_1(h) \equiv 0$. Such a situation appears e.g. in the generic non-Hamiltonian cases after quadratic perturbation of the form $dH/M - \varepsilon\omega = 0$, $\omega = g(x, y)dx - f(x, y)dy = \omega_1 + \omega_2$, provided each of the particular systems $dH/M - \varepsilon\omega_j = 0$ is integrable but the total perturbation is not (in [29] this possibility is not mentioned). Will then the class of perturbations which keep $M_1(h)$ identically zero produce more limit cycles than the class with $M_1(h) \not\equiv 0$ or $M_1(h), M_2(h)$ etc. will give the same upper bound?

We return now to the general case (2). Our first aim in this paper is to select a set of *essential* perturbations which can realize the maximum

number of limit cycles produced by the whole class of systems (2), provided we consider bifurcations of any order in ε . We prove the following result.

THEOREM 1. – *The upper bound for the number of limit cycles produced by the period annulus of a quadratic center under quadratic perturbations (2) is realizable by the following list of essential perturbations (for each one of the cases):*

(i) *Generic Hamiltonian center Q_3^H , $c \neq 0$:*

$$\dot{z} = (\lambda_1 \varepsilon - i) z + (-1 + \lambda_2 \varepsilon + i \lambda_3 \varepsilon) z^2 + 2|z|^2 + (b + ic) \bar{z}^2.$$

(ii) *Generic reversible center Q_3^R , $a \neq -1, 4$ or $a = 4$, $b \neq \pm 2$:*

$$\dot{z} = (\lambda_1 \varepsilon - i) z + (a + i \lambda_3 \varepsilon) z^2 + 2|z|^2 + (b + i \lambda_5 \varepsilon) \bar{z}^2.$$

(iii) *Generic codimension four center Q_4 , $|b + ic| = 2$, $c \neq 0$:*

$$\dot{z} = (\lambda_1 \varepsilon - i) z + (4 + \lambda_2 \varepsilon + i \lambda_3 \varepsilon) z^2 + 2|z|^2 + (b + ic + i \lambda_5 \varepsilon) \bar{z}^2.$$

(iv) *Generic Lotka-Volterra center Q_3^{LV} , $c \neq 0$:*

$$\dot{z} = (\lambda_1 \varepsilon - i) z + z^2 + (\lambda_2 \varepsilon + i \lambda_3 \varepsilon) |z|^2 + (b + ic) \bar{z}^2.$$

(v) *Reversible Hamiltonian center $Q_3^H \cap Q_3^R$:*

$$\dot{z} = (\lambda_1 \varepsilon^2 - i) z + (-1 + \lambda_2 \varepsilon + i \lambda_3 \varepsilon^2) z^2 + 2|z|^2 + (b + i\varepsilon) \bar{z}^2.$$

(vi) *Reversible codimension four center $Q_4 \cap Q_3^R$:*

$$\dot{z} = (\lambda_1 \varepsilon^2 - i) z + (4 + \lambda_2 \varepsilon + i \lambda_3 \varepsilon^2) z^2 + 2|z|^2 + (b + \lambda_4 \varepsilon + i\varepsilon) \bar{z}^2, \quad b = \pm 2.$$

(vii) *Reversible Lotka-Volterra center Q_3^{LV} , $c = 0$:*

$$\dot{z} = (\lambda_1 \varepsilon^2 - i) z + z^2 + (\lambda_2 \varepsilon + i \lambda_3 \varepsilon^2) |z|^2 + (b + i\varepsilon) \bar{z}^2.$$

(viii) *Hamiltonian triangle:*

$$\dot{z} = (\lambda_1 \varepsilon^3 - i) z + (\lambda_2 \varepsilon^2 + i \lambda_3 \varepsilon) z^2 + i \lambda_5 \varepsilon |z|^2 + \bar{z}^2.$$

In the above equations, λ_k are arbitrary real constants independent on ε .

We point out that the perturbations in Theorem 1 are chosen so that in the generic cases (i)-(iv) the displacement function for (2) has the form (4)

where M_1 is the Melnikov function defined by (3). In the degenerate cases, the integral $M_1(h)$ cannot be used as a bifurcation function, as explained above. The reason is that the additional symmetry results in decreasing the number of zeros possessed by $M_1(h)$. Thus, in cases (v)-(vii) the perturbation has to be chosen so that the displacement function will have the form (5) and for the Hamiltonian triangle (viii) it should be

$$d(h, \varepsilon) = \varepsilon^3 M_3(h) + O(\varepsilon^4).$$

As a consequence, in the generic case we only need to consider the Melnikov integral $M_1(h)$ while in the degenerate case we have to investigate the coefficient M_2 and for perturbations of the Hamiltonian triangle we need even the third coefficient M_3 in the expansion of the displacement function. The functions $M_k(h)$, $k \geq 2$ are called sometimes the higher order Melnikov functions.

It should be noted however that such an approach is reasonable only if one wants to determine the *sharp upper bound* for the number of limit cycles within the whole class of quadratic perturbations. For a particular perturbation, the *exact* number of limit cycles (produced by the period annulus) may be lower since it is governed by the first nonzero Melnikov function. Therefore the problem of finding the exact number of the limit cycles for any concrete system requires (in principle) the investigation of Melnikov functions of any order which can be a formidable problem.

Our next goal is to calculate the bifurcation function corresponding to each of the perturbations given in Theorem 1. This procedure requires to know explicitly the corresponding first integral H and integrating factor M . For our convenience, these are given in the appendix at the end of this paper. In this connection, we note that it is somewhat cumbersome to find the first integral for the generic Lotka-Volterra center Q_3^{LV} , $c \neq 0$ in the present coordinates. For this reason we additionally perform in this case a rotation and scaling of coordinates (which is independent of ε). The rotation angle and the scaling are chosen so that in the new coordinates the unperturbed system will have $y = \frac{1}{2}$ as an invariant line. In these coordinates, the equation of the generic Q_3^{LV} is

$$(6) \quad \dot{z} = -iz + \left[1 + \frac{1}{2}(b - ic)\right] z^2 + \frac{1}{2}(b + ic) \bar{z}^2,$$

where we can restrict without loss of generality the parameters to vary in the region $-2 < b < 1$, $0 < c^2 < (1 - b)(2 + b)^2(3 + b)^{-1}$. The

equations of Γ_2 and Γ_∞ become $b^2 + c^2 = 1$ and $b = -1$ respectively. The above change results in a rotation and scaling the vector (λ_2, λ_3) in the perturbed Lotka-Volterra system (iv) whilst λ_1 remains unchanged. Therefore we will preferably consider in what follows

$$(7) \quad \dot{z} = (\lambda_1 \varepsilon - i) z + \left[1 + \frac{1}{2} (b - ic) \right] z^2 + (\lambda_2 \varepsilon + i \lambda_3 \varepsilon) |z|^2 + \frac{1}{2} (b + ic) \bar{z}^2$$

instead of the system in (iv). As for the reversible part of Q_3^{LV} , we keep the former coordinates. The results of our calculations are expressed in the following two theorems.

THEOREM 2 (Bifurcation functions for the generic cases). – *The exact upper bound for the number of limit cycles produced by the period annulus of a generic quadratic center under quadratic perturbations (2) is equal (for each one of the cases) to the maximum number of zeros in (h_c, h_s) , counting multiplicities, of the related Melnikov integral as follows:*

(i) *Generic Hamiltonian center Q_3^H ($c \neq 0$):*

$$M_1(h) = \int \int_{H(x,y) < h} (\mu_1 + \mu_2 x + \mu_3 y) dx dy.$$

(ii) *Generic reversible center Q_3^R ($((a+1)(|a-4| + |b^2-4|) \neq 0)$):*

(1) $a \neq b$, $a \neq 3b+2$,

$$M_1(h) = \int \int_{H(x,y) < h} M(x) (\mu_1 + \mu_2 x + \mu_3 x^{-1}) dx dy.$$

(2) $a \neq b$, $a = 3b+2$,

$$M_1(h) = \int \int_{H(x,y) < h} x^{-3} [\mu_1 + \mu_2 x + \mu_3 x (\kappa \ln x - h)] dx dy.$$

(3) $a = b$, $M_1(h) = \int \int_{H(x,y) < h} M(x) (\mu_1 + \mu_2 x + \mu_3 x^2) dx dy.$

(iii) *Generic codimension four center Q_4 ($c \neq 0$):*

$$M_1(h) = \int \int_{H(x,y) < h} x^{-7/2} [\mu_1 + \mu_2 y + \mu_3 y^3 + \mu_4 (\kappa^2 y^4 - x^2)] dx dy,$$

$$\kappa = 4(b+2)^{-1}.$$

(iv) *Generic Lotka-Volterra center Q_3^{LV} in form (6):*

(1) $b^2 + c^2 > 1$,

$$M_1(h) = \int \int_{H(x,y) < h} M(x, y) (\mu_1 + \mu_2 x^{-1} + \mu_3 y^{-1}) dx dy.$$

$$(2) \ b^2 + c^2 < 1,$$

$$M_1(h) = \int \int_{H(x,y) < h} M(x,y) [\mu_1 + (\mu_2 x + \mu_3 y)(x^2 + y^2)^{-1}] dx dy.$$

$$(3) \ b^2 + c^2 = 1,$$

$$M_1(h) = \int \int_{H(x,y) < h} M(x,y) (\mu_1 + \mu_2 x^{-1} + \mu_3 x^{-2}) dx dy.$$

$$(4) \ b = -1,$$

$$M_1(h) = \int \int_{H(x,y) < h} M(x,y) (\mu_1 + \mu_2 x^{-1} + \mu_3 x^2) dx dy.$$

In the above integrals, μ_j are independent constants which are linear combinations of the former λ_k and in formula (ii)-(2), $\kappa = \frac{9}{16} (a-2)(a+1)^{-3}$.

Of course, the proof of Theorem 2 consists of routine calculations of integrals according to formula (3). We present them in section 3 below. As for the computation of $M_2(h)$ needed in the reversible cases (v)-(vii), we use the following procedure which seems to be very natural. In the case Q_3^R , the first Melnikov function is of the form $M_1(h) = \lambda_1 I_1(h) + \lambda_3 I_3(h) + \lambda_5 I_5(h)$. The integral $I_5(h)$ becomes zero on $Q_3^R \cap (Q_3^H \cup Q_4)$. Then it is easy to deduce (based on the proof of Theorem 1) that

$$M_2(h) = [\lambda_1 I_1(h) + \lambda_3 I_3(h) + \lambda_2 (d/da) I_5(h)]|_{a=-1} \\ \text{in case } Q_3^R \cap Q_3^H,$$

$$M_2(h) = [\lambda_1 I_1(h) + \lambda_3 I_3(h) + \lambda_2 (d/da) I_5(h) \\ + \lambda_4 (d/db) I_5(h)]|_{a=4, |b|=2} \text{ in } Q_3^R \cap Q_4.$$

Similarly, for the Lotka-Volterra center Q_3^{LV} written in the original coordinates, the first Melnikov function is $M_1(h) = \lambda_1 I_1(h) + \lambda_2 I_2(h) + \lambda_3 I_3(h)$ (with some other I_k). The integral $I_2(h)$ vanishes for $c = 0$ and we have accordingly

$$M_2(h) = [\lambda_1 I_1(h) + \lambda_2 (d/dc) I_2(h) + \lambda_3 I_3(h)]|_{c=0} \text{ in } Q_3^{LV}, \\ c = 0.$$

Hence, in the degenerate cases, the new integrals appearing in M_2 are in fact the derivatives with respect to parameters of the vanishing integrals

in M_1 . It should be mentioned that the above functions I_k do not depend on the perturbation. They only depend on the point of the center variety and reflect the geometry of ovals around the center $z = 0$. Computing the corresponding derivatives with respect to parameters, we get

THEOREM 3 (Bifurcation functions for the degenerate cases). – *The exact upper bound for the number of limit cycles produced by the period annulus of a degenerate quadratic center under quadratic perturbations (2) is equal (for each one of the cases) to the maximum number of zeros in (h_c, h_s) , counting multiplicities, of the related Melnikov integral as follows:*

(v) *Reversible Hamiltonian center $Q_3^H \cap Q_3^R$:*

$$(1) \ b \neq -1, \ M_2(h) = \int \int_{H(x,y) < h} (\mu_1 + \mu_2 x + \mu_3 x^{-1}) dx dy.$$

$$(2) \ b = -1, \ M_2(h) = \int \int_{H(x,y) < h} (\mu_1 + \mu_2 x + \mu_3 h) dx dy.$$

(vi) *Reversible codimension four center $Q_4 \cap Q_3^R$:*

$$(1) \ b = 2,$$

$$M_2(h) = \int \int_{H(x,y) < h} M(x) \times [\mu_1 + \mu_2 x + \mu_3 x^{-2} + \mu_4 (1 - x^{-1}) \ln x] dx dy.$$

$$(2) \ b = -2,$$

$$M_2(h) = \int \int_{H(x,y) < h} M(x) \times [\mu_1 + \mu_2 x + \mu_3 x^2 + \mu_4 (2x - 1 - x^{-1}) \ln x] dx dy.$$

(vii) *Reversible Lotka-Volterra center Q_3^{LV} ($c = 0$):*

$$(1) \ b = -1, \ M_2(h) = \int \int_{H(x,y) < h} M(x) (\mu_1 + \mu_2 x^{-1} + \mu_3 x^2) dx dy.$$

$$(2) \ b = 0,$$

$$M_2(h) = \int_{H(x,y)=h} M(x) [\mu_1 y + \mu_2 x^{-1} y + \mu_3 (x^2 - x) y^{-1} \ln x] dx.$$

$$(3) \ b = \frac{1}{3}, \ M_2(h) = \int \int_{H(x,y) < h} M(x) (\mu_1 + \mu_2 x^{-1} + \mu_3 x^{-2}) dx dy.$$

$$(4) \ b = 1, \ M_2(h) = \int_{H(x,y)=h} M(x) (\mu_1 y + \mu_2 x^2 y + \mu_3 xy^{-1}) dx.$$

(5) *General case,*

$$M_2(h) = \int_{H(x,y)=h} M(x) [\mu_1 y + \mu_2 x^{-1} y + \mu_3 (x-1) y^{-1}] dx.$$

(viii) *Hamiltonian triangle*:

$$M_3(h) = \int \int_{H(x,y) < h} [\mu_1 + \mu_2 x^{-1} + \mu_3 h^{-1} + \mu_4 h^{-1} (x-1) \ln x] dx dy.$$

Except for (viii), in the above integrals, μ_j are independent constants which are linear combinations of the former λ_k . In (viii), μ_j are cubic polynomials of λ_k .

It is perhaps interesting to note that formulas (iv)-(4), (v)-(2) and (vii)-(2) above present the bifurcation function for the standard Lotka-Volterra system, for the Bogdanov-Takens system and for the isochronous center S_1 , respectively.

The estimating the number of zeros of the integrals listed above is in general a difficult problem. For this reason definitive results are known for only few of them. The facts accumulated in several papers up to now say that most of the bifurcation functions above will obey on (h_c, h_s) the Chebyshev's property. Namely, the maximal number of possible zeros equals the number of independent functions included in the corresponding integral minus one. The number of μ_j 's for each of the cases in Theorems 2, 3 as well as the known results about limit cycles in perturbations of quadratic centers we already mentioned suggest that the following conjecture is reasonable (cf. [28]):

CONJECTURE. – *The space of bifurcation functions corresponding to quadratic centers is Chebyshev's in (h_c, h_s) outside the shaded area A_3 in the bifurcation diagram of Q_3^R and Chebyshev's with accuracy 1 inside A_3 . That is, in a quadratic integrable system the cyclicity of a period annulus under quadratic perturbations equals the cyclicity of the center inside, except for centers from the set $A_3 \subset Q_3^R$ (see Fig. 4).*

Recall that the cyclicity of the center $z = 0$ in (1) under quadratic perturbations is 3 for Q_4 , $Q_3^R \cap \{a = 4\}$ and for the Hamiltonian triangle and it is 2 otherwise [1], [28]. Hence, it is conjectured above that the cyclicity of the annulus is 3 for Q_4 , for the interior of the shaded area $A_3 \subset Q_3^R$ and for the Hamiltonian triangle and it is 2 otherwise.

In addition to the results already cited, nowadays it is established that the cyclicity of annuli corresponding to centers inside A_3 should be at least three [20]. In the parameter space, the subset $A_3 \subset Q_3^R$ of centers having (conjectured) cyclicity 3 period annuli is bounded by the line $a = 4$

(of cyclicity 3 centers) and by the curve \mathcal{C}_3 of cyclicity 3 contours on the Poincaré sphere, see Figure 4. As noted in [20], \mathcal{C}_3 is composed of a portion of a straight line and of a winding curve. The latter was not sharply located in [20]. In the next theorem we give the exact formula of the winding part of \mathcal{C}_3 . It turns out to be not an elementary curve in the (a, b) -plane and can probably be localized by computer calculations only. To formulate the statement, we need to introduce some notation. Throughout the paper, we let denote

$$(8) \quad J_k(h) = \int_{H(x,y)=h} M(x) x^k y \, dx = - \int \int_{H(x,y)<h} M(x) x^k \, dx \, dy$$

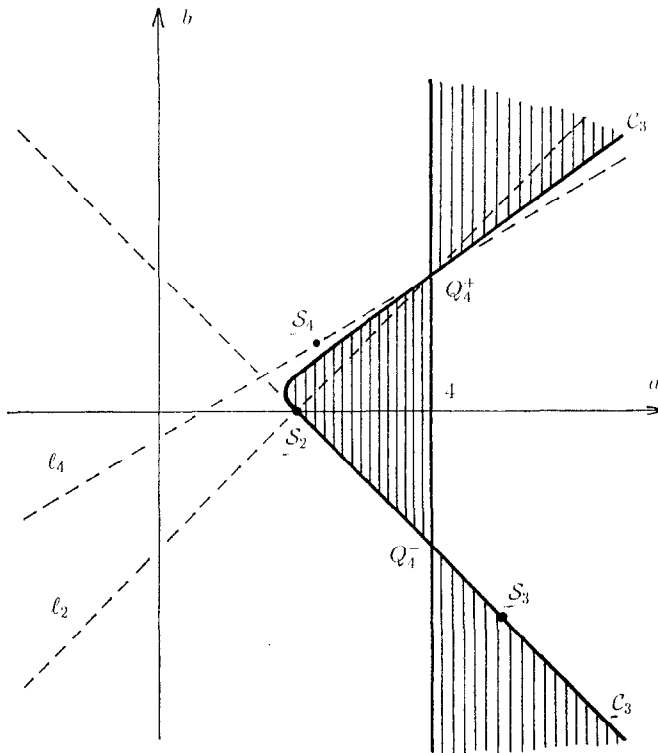


Fig. 4. – Location of the curve \mathcal{C}_3 in the bifurcation diagram of Q_3^R in the (a, b) -plane. The shaded area corresponds to systems for which the center at the origin has cyclicity three period annulus.

(whenever the integrals exist) and let $W[J_i, J_k](h) = J_i(h)J'_k(h) - J'_i(h)J_k(h)$ be the corresponding Wronskian function.

THEOREM 4. – *In the bifurcation diagram of Q_3^R in the (a, b) -plane, the curve \mathcal{C}_3 is formed by the half-line $a + b - 2 = 0$, $b \leq 0$ and the winding line $W[J_0, J_{-1}](0) = 0$, $b > 0$.*

It should be noted that the curve \mathcal{C}_3 intersects the line $a = 4$ at points Q_4^\pm . Moreover \mathcal{C}_3 goes through the isochronous centers $\mathcal{S}_2, \mathcal{S}_3$ [20].

We next apply two of the formulas derived in Theorem 3 to determine the cyclicity of the period annulus for two particular cases: the isochronous center \mathcal{S}_1 and one of the points in $Q_4 \cap Q_3^R$. They both require a second order analysis in ε .

THEOREM 5. – *The cyclicity of the period annulus of the reversible codimension four center $Q_4^+ = \{Q_4 \cap Q_3^R, b = 2\}$ is equal to three.*

THEOREM 6. – *The cyclicity of the period annulus of the isochronous center \mathcal{S}_1 is two.*

Theorems 4, 5 and 6 are proved in Section 5. We obtain the proof of Theorem 4 by inspecting the asymptotic expansion of integrals at the ends of the interval (h_c, h_s) . The proof of Theorem 5 relies upon the fact that the integrals included in M_2 satisfy Picard-Fuchs system of size three. Using it we are able to derive a scalar first order linear equation satisfied by the bifurcation function M_2 itself. This allows to complete easily the proof. It should be noted that the centers from a small open neighborhood of Q_4^+ (in the center variety) have period annuli which cyclicity cannot exceed the cyclicity of the annulus of Q_4^+ . Henceforth the result in Theorem 5 implies that the same upper bound (three) holds for small portions of Q_3^R and Q_4 near their intersection point Q_4^+ . The proof of Theorem 6 is almost straightforward since the periodic orbits around \mathcal{S}_1 are conic ovals.

2. Essential perturbations

Below we prove Theorem 1 making use of Bautin's fundamental lemma. In consequence of the perturbation (2), the center at the origin becomes a focus and moves slightly. Translate this focus back to the origin and perform a linear change of variables to express the linear part of the system

in a normal form for foci (it depends analytically on ε). Then, in complex coordinates, the system (2) reads

$$(9) \quad \dot{z} = (\lambda_1(\varepsilon) - i)z + A(\varepsilon)z^2 + B(\varepsilon)|z|^2 + C(\varepsilon)\bar{z}^2$$

with λ_1, A, B, C analytic in ε . Accordingly to the three cases $B(0) = 2$; $B(0) = 0, A(0) = 1$; $B(0) = A(0) = 0, C(0) = 1$, further analytic change in (9) (which also is close to the identical one) removes the perturbation in the coefficients B, A or C respectively. Thus we can write an arbitrary perturbation (2) in one of the forms

$$(10) \quad \dot{z} = [\lambda_1(\varepsilon) - i]z + [a + \lambda_2(\varepsilon) + i\lambda_3(\varepsilon)]z^2 + 2|z|^2 + [b + ic + \lambda_4(\varepsilon) + i\lambda_5(\varepsilon)]\bar{z}^2$$

where $a = -1$ or $c = 0$ or $a = 4, |b + ic| = 2$,

$$(11) \quad \dot{z} = [\lambda_1(\varepsilon) - i]z + z^2 + [\lambda_2(\varepsilon) + i\lambda_3(\varepsilon)]|z|^2 + [b + ic + \lambda_4(\varepsilon) + i\lambda_5(\varepsilon)]\bar{z}^2,$$

$$(12) \quad \dot{z} = [\lambda_1(\varepsilon) - i]z + [\lambda_2(\varepsilon) + i\lambda_3(\varepsilon)]z^2 + [\lambda_4(\varepsilon) + i\lambda_5(\varepsilon)]|z|^2 + \bar{z}^2.$$

In the above equations, the bifurcation parameters $\lambda_k(\varepsilon)$ are analytical for small ε , with $\lambda_k(0) = 0$. Write $\lambda_k(\varepsilon) = \sum_{j=1}^{\infty} \lambda_{kj} \varepsilon^j$ and set for short $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$.

Given an energy level $h \in (h_c, h_s)$, denote by $(\xi, 0)$, $\xi = \xi(h) > 0$ the intersection point of the oval from the period annulus contained in the level curve $H(x, y) = h$ with the abscissa. Clearly, $h - h_c = H(\xi, 0) - H(0, 0) \sim \frac{1}{2} M(0, 0) \xi^2$ as $\xi \rightarrow 0$. Let $(\mathcal{P}(\xi, \varepsilon), 0)$ be the point determined by the first return map. It has been proved in [1], [28] that the displacement function for (10)-(12) $d(\xi, \varepsilon) = \mathcal{P}(\xi, \varepsilon) - \xi$ has the following local representation near $\lambda = \xi = 0$

$$d(\xi, \varepsilon) = \lambda_1 \sum_{k=1}^{\infty} \delta_k(\lambda) \xi^k + \bar{v}_3 \xi^3 \left(1 + \sum_{k=1}^{\infty} \alpha_k(\lambda) \xi^k \right) + \bar{v}_5 \xi^5 \left(1 + \sum_{k=1}^{\infty} \beta_k(\lambda) \xi^k \right) + \bar{v}_7 \xi^7 \left(1 + \sum_{k=1}^{\infty} \gamma_k(\lambda) \xi^k \right),$$

where $\delta_1(\lambda) = \frac{1}{\lambda_1} [\exp(2\pi\lambda_1) - 1]$. Moreover, δ_k , α_k , β_k and γ_k are analytic in λ and hence in ε . If we define, following CHICONE and JACOBS [2], [3],

$$\begin{aligned}\delta(\xi) &= \left(2\pi + \sum_{k=2}^{\infty} \delta_k(0) \xi^{k-1}\right) \xi, & \alpha(\xi) &= \left(1 + \sum_{k=1}^{\infty} \alpha_k(0) \xi^k\right) \xi^3, \\ \beta(\xi) &= \left(1 + \sum_{k=1}^{\infty} \beta_k(0) \xi^k\right) \xi^5, & \gamma(\xi) &= \left(1 + \sum_{k=1}^{\infty} \gamma_k(0) \xi^k\right) \xi^7,\end{aligned}$$

then the following representation of the displacement function holds near $\xi = \varepsilon = 0$:

$$\begin{aligned}d(\xi, \varepsilon) &= \lambda_1 [\delta(\xi) + O(\lambda)] + \bar{v}_3 [\alpha(\xi) + O(\lambda)] \\ &\quad + \bar{v}_5 [\beta(\xi) + O(\lambda)] + \bar{v}_7 [\gamma(\xi) + O(\lambda)].\end{aligned}$$

Let us note that the functions α, β, γ and δ do not depend on the perturbation. Next we use the explicit formulas for the Lyapunov quantities \bar{v}_j (taken with minus signs since the normal form we use has a different linear part; cf. ŻOŁĄDEK [28, formula (12)]). Considering first the equation (10), we have

$$\begin{aligned}\bar{v}_3 &= 2\pi \operatorname{Im} AB = 4\pi \lambda_3, \\ \bar{v}_5 &= \frac{2}{3} \pi \operatorname{Im} [(2A + \bar{B})(A - 2\bar{B})\bar{B}C] \\ &= \frac{8}{3} \pi [(a + 1 + \lambda_2)(a - 4 + \lambda_2)(c + \lambda_5) + (2a - 3)b\lambda_3] + \text{h.o.t.}, \\ \bar{v}_7 &= \frac{5}{4} \pi (|B|^2 - |C|^2) \operatorname{Im} [(2A + \bar{B})\bar{B}^2 C] \\ &= 10\pi [4 - (b + \lambda_4)^2 - (c + \lambda_5)^2] (a + 1 + \lambda_2)(c + \lambda_5) \\ &\quad + 10\pi (4 - b^2 - c^2)b\lambda_3 + \text{h.o.t.}\end{aligned}$$

Here and below, by h.o.t. we denote the terms which for small ε are negligible compared with the written terms in \bar{v}_j (in fact, we have omitted above just the terms which are negligible compared with λ_3). In a similar way we get for (11) and (12), respectively,

$$\begin{aligned}\bar{v}_3 &= 2\pi \lambda_3 \\ \bar{v}_5 &= \frac{4}{3} \pi (c\lambda_2 - b\lambda_3 + \lambda_2\lambda_5) + \text{h.o.t.} \\ \bar{v}_7 &= \text{h.o.t.}\end{aligned}$$

and

$$\begin{aligned}
 \bar{v}_3 &= 2\pi(\lambda_3\lambda_4 + \lambda_2\lambda_5) \\
 (13) \quad \bar{v}_5 &= \frac{2}{3}\pi\lambda_5[(2\lambda_3^2 - \lambda_3\lambda_5) + 2(3\lambda_4^2 - \lambda_5^2 + 2\lambda_3\lambda_5)] + \text{h.o.t.}, \\
 \bar{v}_7 &= \frac{5}{4}\pi\lambda_5(3\lambda_4^2 - \lambda_5^2 + 2\lambda_3\lambda_5) + \text{h.o.t.}
 \end{aligned}$$

Thus we obtain for any of the possible cases that, modulo higher order terms, the following expressions for \bar{v}_3 , \bar{v}_5 and \bar{v}_7 hold:

$$\begin{aligned}
 \text{(i) } Q_3^H \setminus Q_3^R &\Rightarrow \bar{v}_3 = 4\pi\lambda_3, \quad \bar{v}_5 = -\frac{40}{3}\pi(c\lambda_2 + b\lambda_3), \\
 &\quad \bar{v}_7 = 10\pi(4 - b^2 - c^2)(c\lambda_2 + b\lambda_3). \\
 \text{(ii) } Q_3^R \setminus (Q_3^H \cup Q_4) &\Rightarrow \bar{v}_3 = 4\pi\lambda_3, \\
 &\quad \bar{v}_5 = \frac{8}{3}\pi[(a^2 - 3a - 4)\lambda_5 + (2a - 3)b\lambda_3], \\
 &\quad \bar{v}_7 = 10\pi(4 - b^2)(b\lambda_3 + (a + 1)\lambda_5). \\
 \text{(iii) } Q_4 \setminus Q_3^R &\Rightarrow \bar{v}_3 = 4\pi\lambda_3, \quad \bar{v}_5 = \frac{40}{3}\pi(c\lambda_2 + b\lambda_3), \\
 &\quad \bar{v}_7 = -50\pi(2bc\lambda_4 + 2c^2\lambda_5 + c\lambda_4^2). \\
 \text{(iv) } Q_3^{LV}, \quad c \neq 0 &\Rightarrow \bar{v}_3 = 2\pi\lambda_3, \quad \bar{v}_5 = \frac{4}{3}\pi(c\lambda_2 - b\lambda_3), \quad \bar{v}_7 = 0. \\
 \text{(v) } Q_3^H \cap Q_3^R &\Rightarrow \bar{v}_3 = 4\pi\lambda_3, \quad \bar{v}_5 = -\frac{40}{3}\pi(b\lambda_3 + \lambda_2\lambda_5), \\
 &\quad \bar{v}_7 = 10\pi(4 - b^2)(b\lambda_3 + \lambda_2\lambda_5). \\
 \text{(vi) } Q_4 \cap Q_3^R &\Rightarrow \bar{v}_3 = 4\pi\lambda_3, \quad \bar{v}_5 = \frac{40}{3}\pi(b\lambda_3 + \lambda_2\lambda_5), \\
 &\quad \bar{v}_7 = -50\pi(2b\lambda_4\lambda_5 + \lambda_5^3). \\
 \text{(vii) } Q_3^{LV}, \quad c = 0 &\Rightarrow \bar{v}_3 = 2\pi\lambda_3, \quad \bar{v}_5 = -\frac{4}{3}\pi(b\lambda_3 - \lambda_2\lambda_5), \quad \bar{v}_7 = 0.
 \end{aligned}$$

and for the Hamiltonian triangle see (13).

Below we consider in more details the Hamiltonian case Q_3^H . The remaining cases can be treated similarly. By (i) above, in the generic Hamiltonian case the displacement function for (10) has the form

$$d(\xi, \varepsilon) = L_1(\xi) \lambda_1(\varepsilon) + L_2(\xi) \lambda_2(\varepsilon) + L_3(\xi) \lambda_3(\varepsilon) + \text{h.o.t.}$$

where $L_k(\xi)$ are analytic and linearly independent functions, presented for small ξ respectively by the convergent power series

$$l_1(\xi) = \delta(\xi),$$

$$l_2(\xi) = -\frac{40}{3} \pi c \beta(\xi) + 10 \pi (4 - b^2 - c^2) c \gamma(\xi),$$

$$l_3(\xi) = 4 \pi \alpha(\xi) - \frac{40}{3} \pi b \beta(\xi) + 10 \pi (4 - b^2 - c^2) b \gamma(\xi).$$

If we express now $d(\xi, \varepsilon)$ in a power series: $d(\xi, \varepsilon) = \sum_{k=1}^{\infty} d_k(\xi) \varepsilon^k$, then one obtains immediately that the first nonzero coefficient (say d_k) in the expansion of the displacement function is given by

$$d_k(\xi) = L_1(\xi) \lambda_{1k} + L_2(\xi) \lambda_{2k} + L_3(\xi) \lambda_{3k}.$$

Therefore we can take $\lambda_j(\varepsilon) = \lambda_j \varepsilon$, $j = 1, 2, 3$ and $\lambda_4(\varepsilon) = \lambda_5(\varepsilon) = 0$ in order to obtain the most general linear combination of $L_j(\xi)$ and respectively to produce a perturbation possessing the possible maximal number of limit cycles.

For the reversible Hamiltonian case (v), the displacement function is

$$d(\xi, \varepsilon) = L_1(\xi) \lambda_1(\varepsilon) + L_2(\xi) \lambda_2(\varepsilon) \lambda_5(\varepsilon) + L_3(\xi) \lambda_3(\varepsilon) + \text{h.o.t.}$$

where $L_1(\xi)$ is as above and L_2, L_3 are analytic functions, presented for small ξ by the following convergent power series

$$l_2(\xi) = -\frac{40}{3} \pi \beta(\xi) + 10 \pi (4 - b^2) \gamma(\xi),$$

$$l_3(\xi) = 4 \pi \alpha(\xi) - \frac{40}{3} \pi b \beta(\xi) + 10 \pi (4 - b^2) b \gamma(\xi).$$

In this situation d_1 has the form $d_1(\xi) = L_1(\xi)\lambda_{11} + L_3(\xi)\lambda_{31}$ and if $d_1(\xi) = \dots = d_{k-1}(\xi) \equiv 0$, then

$$(14) \quad d_k(\xi) = L_1(\xi)\lambda_{1k} + L_2(\xi) \left(\sum_{i+j=k} \lambda_{2i}\lambda_{5j} \right) + L_3(\xi)\lambda_{3k}$$

(note that in fact the sum at L_2 contains just one term). Hence d_1 can never include three linearly independent functions. To obtain a perturbation producing the possible maximum of limit cycles, we have to choose λ_j so that $d_1(\xi) \equiv 0$ and moreover, in the expression (14) for $d_2(\xi)$, the coefficients at L_j to be independent. For this purpose, we can take $\lambda_1(\varepsilon) = \lambda_1 \varepsilon^2$, $\lambda_2(\varepsilon) = \lambda_2 \varepsilon$, $\lambda_3(\varepsilon) = \lambda_3 \varepsilon^2$, $\lambda_4(\varepsilon) = 0$ and $\lambda_5(\varepsilon) = \varepsilon$ thus proving (v). The remaining cases in (i)-(viii) are considered in a quite similar way and we omit the details. Theorem 1 is proved.

It should be noticed that the formulas derived within the proof provide a simple algorithm for computing the second Melnikov function required in the degenerate cases. We summarize these which we will need hereafter in the following corollary.

COROLLARY 1. – *For the list of essential perturbations as given in Theorem 1, the first nonzero coefficient $d_k(\xi)$ in the expansion of the displacement function $d(\xi, \varepsilon)$ is given respectively by the formula:*

$$\begin{aligned} \text{(i)} \quad Q_3^H \setminus Q_3^R &\Rightarrow d_1 = \lambda_1 L_1^H + \lambda_2 L_2^H + \lambda_3 L_3^H, \\ \text{(ii)} \quad Q_3^R \setminus (Q_3^H \cup Q_4) &\Rightarrow d_1 = \lambda_1 L_1^R + \lambda_3 L_3^R + \lambda_5 L_5^R, \\ \text{(iv)} \quad Q_3^{LV}, c \neq 0 &\Rightarrow d_1 = \lambda_1 L_1^{LV} + \lambda_2 L_2^{LV} + \lambda_3 L_3^{LV}, \\ \text{(v)} \quad Q_3^H \cap Q_3^R &\Rightarrow d_2 = [\lambda_1 L_1^H + \lambda_2 \partial_c L_2^H + \lambda_3 L_3^H]|_{c=0} \\ &= [\lambda_1 L_1^R + \lambda_2 \partial_a L_5^R + \lambda_3 L_3^R]|_{a=-1}, \\ \text{(vi)} \quad Q_4 \cap Q_3^R &\Rightarrow d_2 = [\lambda_1 L_1^R + \lambda_3 L_3^R + \lambda_2 \partial_a L_5^R \\ &\quad + \lambda_4 \partial_b L_5^R]|_{a=4, |b|=2}, \\ \text{(vii)} \quad Q_3^{LV}, c = 0 &\Rightarrow d_2 = [\lambda_1 L_1^{LV} + \lambda_2 \partial_c L_2^{LV} + \lambda_3 L_3^{LV}]|_{c=0}. \end{aligned}$$

In these formulas, L_j^H , L_j^R and L_j^{LV} are analytic functions of ξ , independent on the perturbation. They are defined through specific linear combinations of $\alpha, \beta, \gamma, \delta$.

The proof of Corollary 1 is immediate and for this reason will be omitted. Now, we have $d(h, \varepsilon) = H(\mathcal{P}(\xi, \varepsilon), 0) - H(\xi, 0) = d(\xi, \varepsilon) H_x(\xi^*, 0)$ where $\xi = \xi(h)$ is the solution of $H(\xi, 0) = h$. Clearly $H_x(\xi, 0) \neq 0$ because the ovals intersect the abscissa transversally. Therefore $M_j(h) = d_j(\xi) H_x(\xi, 0)$, $j = 1, 2, \dots$. Hence, by Corollary 1, if $M_1(h) = \lambda_1 I_1(h) + \lambda_2 I_2(h) + \lambda_3 I_3(h)$ is the Melnikov function for the Hamiltonian case (i), then the second Melnikov function needed for the degenerate case (v) is $M_2(h) = [\lambda_1 I_1(h) + \lambda_2 (d/dc) I_2(h) + \lambda_3 I_3(h)]|_{c=0}$ (we use the equality $L_2^H|_{c=0} = 0$). The other formulas outlined in the introduction are obtained similarly.

3. Bifurcation functions for the generic cases

In this section we prove Theorem 2. The statement in (i) is obvious. As for the remaining cases, the main part of the proof consists of routine but long calculations. Below we give them in more details. The calculations are always preceded by a rotation of the coordinates on angle $\pi/2$ which aims at obtaining the final formulas in coordinates that had been at most often used before. Thus in the reversible cases, the axis of symmetry becomes $y = 0$, as usual.

(ii) System (ii), rotated on an angle $\pi/2$, in real variables (x, y) reads

$$\begin{aligned}\dot{x} &= y + (2a - 2b)xy + \varepsilon f_0(x, y) \\ \dot{y} &= -x + (2 - a - b)x^2 + (2 + a + b)y^2 + \varepsilon g_0(x, y),\end{aligned}$$

with $f_0 = \lambda_1 x + (\lambda_3 + \lambda_5)(x^2 - y^2)$, $g_0 = \lambda_1 y + (2\lambda_3 - 2\lambda_5)xy$. The first integral has the form (see the Appendix) $H = XM(\frac{1}{2}y^2 - U)$, where $X = 1 + (2a - 2b)x$. The integrating factor M and U are functions of x . We note that $M_x = -4(a + 1)MX^{-1}$. Then by (3),

$$\begin{aligned}(15) M_1(h) &= - \int \int_{H < h} [M(f_{0x} + g_{0y}) + M_x f_0] dx dy \\ &= - \int \int_{H < h} M(2\lambda_1 + 4\lambda_3 x) dx dy + 4(a + 1) \\ &\quad \times \int \int_{H < h} MX^{-1} [\lambda_1 x + (\lambda_3 + \lambda_5)(x^2 - y^2)] dx dy.\end{aligned}$$

To eliminate the term containing y^2 , we integrate the equation $H = h$ rewritten as $M(\frac{1}{2}y^2 - U) = hX^{-1}$:

$$\begin{aligned}\int_{H=h} M\left(\frac{1}{2}y^2 - U\right) dy &= \int_{H=h} hX^{-1} dy \\ &= (2a - 2b) \int_{H=h} hX^{-2} y dx \\ &= (2a - 2b) \int_{H=h} MX^{-1} \left(\frac{1}{2}y^2 - U\right) y dx.\end{aligned}$$

Applying Stokes' formula to both sides of this formula yields

$$(16) \quad \iint_{H < h} MX^{-1} [(a - 3b - 2)y^2 + 2(a + b + 2)U - XU_x] dx dy = 0.$$

Using the equation

$$\begin{aligned}\dot{y} &= -H_x/M = XU_x - \left(\frac{1}{2}y^2 - U\right)(MX)_x/M \\ &\equiv -x + (2 - a - b)x^2 + (2 + a + b)y^2\end{aligned}$$

we see that $XU_x - 2(a + b + 2)U = -x + (2 - a - b)x^2$. Replacing in (16) we get the relation

$$(17) \quad \iint_{H < h} MX^{-1} [(a - 3b - 2)y^2 + x + (a + b - 2)x^2] dx dy = 0.$$

Provided $a \neq 3b + 2$, we use (17) to eliminate in (15) the integral containing y^2 . Thus we get (ii)-(3) if $a = b$ and (ii)-(1) otherwise (taking in the latter case X as the new variable). For later use, we give the explicit expression obtained for the general case (ii)-(1):

$$\begin{aligned}(18) \quad M_1 &= \lambda_1 I_1 + \lambda_3 I_3 + \lambda_5 I_5 = \lambda_1 \frac{(a + 1)J_{-1} - (b + 1)J_0}{(a - b)^2} \\ &+ \lambda_3 \frac{\left\{ \begin{aligned} &2(a + 1)J_{-1} + (3ab - 3b^2 - a - 3b - 4)J_0 \\ &- (3ab - 3b^2 - 3b + a - 2)J_1 \end{aligned} \right\}}{(a - b)^3(a - 3b - 2)} \\ &+ \lambda_5 \frac{\left\{ \begin{aligned} &2(a + 1)J_{-1} + (a + 1)(a - b - 4)J_0 \\ &- (a + 1)(a - b - 2)J_1 \end{aligned} \right\}}{(a - b)^3(a - 3b - 2)}.\end{aligned}$$

Recall that the functions $J_k(h)$ are defined by (8).

In the case when $a - 3b - 2 = 0$, along with (17) we derive another relation, simply integrating the equation $MX^{-1}y(\frac{1}{2}y^2 - U) = hMXy$. Together with (17), this yields

$$\begin{aligned} & \int \int_{H < h} M [3X^{-1} + a - 5 - (a - 2)X] dx dy \\ &= \int \int_{H < h} M \left[X^{-1} \left(\frac{3}{2}y^2 - U \right) - hX \right] dx dy = 0. \end{aligned}$$

Taking in (15) X as the new variable, it remains to use the last two equations to exclude from (15) the terms containing X^{-1} and y^2 . In this way we obtain (ii)-(2) with $\kappa = \frac{9}{16}(a - 2)(a + 1)^{-3}$.

(iii) System (iii), rotated on an angle $\pi/2$, becomes

$$\begin{aligned} \dot{x} &= y + c(x^2 - y^2) + (8 - 2b)xy + \varepsilon f_0(x, y) \\ \dot{y} &= -x - (2 + b)x^2 + (6 + b)y^2 - 2cxy + \varepsilon g_0(x, y), \end{aligned}$$

with $f_0 = \lambda_1 x + (\lambda_3 + \lambda_5)(x^2 - y^2) + 2\lambda_2 xy$, $g_0 = \lambda_1 y + (2\lambda_3 - 2\lambda_5)xy + \lambda_2(x^2 - y^2)$. The integrating factor is $M = X^{-5/2}$ (see the Appendix for the definition of X, Y in the considered case). Denote for short $I_{j,k} = \int \int_{H < h} MX^j Y^k dX dY$. A straightforward calculation of $M_1(h)$ followed by a change of variables $(x, y) \rightarrow (X, Y)$ in the integrals yields the formula $M_1(h) = -\frac{1}{8}|c|^{-1}[\lambda_1 F_1 + \lambda_2 F_2 + \lambda_3 F_3 + \lambda_5 F_5]$ where

$$\begin{aligned} F_1 &= \frac{1}{2}[5I_{-1,0} - I_{0,0} - 5\kappa I_{-1,2}], \quad \kappa = 4(b + 2)^{-1}, \\ F_2 &= \frac{5}{2}c^{-1}\kappa^{-1} \left[\frac{1}{2}\kappa(b + 1)J + K + I_{-1,0} - 2I_{0,0} - \kappa(b + 3)I_{0,1} \right. \\ &\quad \left. + 6\kappa I_{-1,2} + \frac{1}{2}\kappa(b + 5)(I_{-1,1} + 2\kappa I_{-1,3}) \right], \\ F_3 &= \frac{5}{2}c^{-2}\kappa^{-1} \left[2J + (b + 8) \left(\frac{1}{5}K + 2\kappa I_{-1,2} \right) + bI_{-1,0} \right. \\ &\quad \left. + \frac{2}{5}(7b - 4)I_{0,0} - 12I_{0,1} + 2I_{-1,1} + 20\kappa I_{-1,3} \right], \\ F_5 &= \frac{5}{2}c^{-2}\kappa^{-1} \left[\frac{1}{2}\kappa(b - 1)J + b(K + I_{-1,0} - 2I_{0,0}) - 4(b + 1)I_{0,1} \right. \\ &\quad \left. + 2\kappa(3b + 4)I_{-1,2} + 2(b + 3)(I_{-1,1} + 2\kappa I_{-1,3}) \right], \\ J &= \kappa^2 I_{-1,5} - 2\kappa I_{0,3} + I_{1,1}, \quad K = 5\kappa^2 I_{-1,4} - 6\kappa I_{0,2} + I_{1,0}. \end{aligned}$$

Next, we multiply the equation $H = h$ by $X^j Y^k$ and integrate along the oval. Then applying Green's formula gives the relation

$$\begin{aligned} & \kappa(2j+k+2)I_{j+1,k+2} + \kappa(4j+3k+4)I_{j+1,k+1} \\ & + (2j+3k+2)I_{j+1,k} \\ & - (2j+k+2)I_{j+2,k} - kI_{j+2,k-1} + kI_{j+1,k-1} = 0. \end{aligned}$$

Using this relation with $j = -1, k = 0, 1$ and $j = -2, 0 \leq k \leq 3$, we get the expressions

$$\begin{aligned} I_{0,0} &= I_{-1,0} + \kappa I_{-1,1}, \quad I_{0,1} = \kappa I_{-1,3} - I_{-1,1}, \quad I_{-1,2} = -I_{-1,1}, \\ K &= 5(\kappa^2 I_{-1,4} - I_{1,0}) + 6I_{-1,0} - 6\kappa I_{-1,1} + 12\kappa^2 I_{-1,3}, \\ J &= -K + I_{-1,0} + (4\kappa - 3)I_{-1,1} - 4\kappa I_{-1,3}. \end{aligned}$$

Replacing these expressions in F_j and then in M_1 , we obtain the formula from (iii).

(iv) Performing a rotation on an angle $\pi/2$ of the perturbed Lotka-Volterra equation in form (7), we obtain in real coordinates the system

$$\begin{aligned} \dot{x} &= y + 2xy + \varepsilon f_0(x, y) \\ \dot{y} &= -x - (b+1)(x^2 - y^2) - 2cxy + \varepsilon g_0(x, y), \end{aligned}$$

where $f_0 = \lambda_1 x - \lambda_3(x^2 + y^2)$ and $g_0 = \lambda_1 y + \lambda_2(x^2 + y^2)$. We recall that in the present coordinates, it suffices to restrict the parameters within the set $\{-2 < b < 1, c \neq 0, b^2 + c^2 < 4(b+3)^{-1}\}$. Then we have

$$\begin{aligned} M_1(h) &= - \int \int_{H < h} [(M f_0)_x + (M g_0)_y] dx dy \\ &= - \int \int [\lambda_1(2M + xM_x + yM_y) + \lambda_2(2yM + (x^2 + y^2)M_y) \\ &\quad - \lambda_3(2xM + (x^2 + y^2)M_x)] dx dy. \end{aligned}$$

Put $w = w(x, y) = (1 + (b+1)x + cy)^2 + (1 - b^2 - c^2)y^2$. Then the equation of the ovals from the period annulus around $x = y = 0$ reads (see the appendix) $H = (1 + 2x)Mw/2(b-1) = h$. Further, one easily obtains (using e.g. the unperturbed system) that

$$\begin{aligned} M_x &= -2[(b+3)cy + (b+2)(1 + (b+1)x)]M/w, \\ M_y &= 2[(b^2 + b - 2)y - c(1 + (b+1)x)]M/w. \end{aligned}$$

Using these relations and taking into account that

$$\begin{aligned} \int \int_{H < h} (M_y w + M w_y) dx dy &= - \int_{H=h} M w dx = 2h(1-b) \\ &\quad \times \int_{H=h} (1+2x)^{-1} dx = 0, \\ \int_{H=h} M(1+2x)(w_x dx + w_y dy) &= \int_{H=h} M(1+2x) dw = 2h(b-1) \\ &\quad \times \int_{H=h} w^{-1} dw = 0, \end{aligned}$$

we get

$$(19) \quad \int \int_{H < h} M x dx dy = \int \int_{H < h} M y dx dy = 0.$$

Consider first the case $b+1 \neq 0$. We have to express the terms in M_1 involving $x^2 + y^2$ in an alternative form. For this, we use first that $\int_{H=h} M(1+2x) w d(w_x w^{-1}) = \int_{H=h} 2h(b-1) d(w_x w^{-1}) = 0$. Evaluating the integral on the left yields the identity

$$(20) \quad \int \int_{H < h} M w^{-1} [cx^2 - cy^2 - 2bxy - y - c(b+1)^{-1}x] dx dy = 0.$$

Similarly, as a consequence of $\int_{H=h} M(1+2x) w d(w_y w^{-1}) = 0$ we get

$$(21) \quad \int \int M w^{-1} [(1-b^2-2c^2)(x^2 + (b+1)^{-1}x) + (1-b^2)y^2 + 2bcxy + cy] dx dy = 0$$

as well as the identities $\int_{H=h} M(1+2x) w d(w_x^2 w^{-1}) = \int_{H=h} M(1+2x) w d(w_y^2 w^{-1}) = 0$ yield respectively

$$(22) \quad \begin{cases} \int \int_{H < h} M w^{-1} xy w_x dx dy = 0, \\ \int \int_{H < h} M w^{-1} x [1 + (b+1)x] w_y dx dy = 0. \end{cases}$$

The next step is to express $(x^2 + y^2) M_x$ and $(x^2 + y^2) M_y$ as linear combinations of $x M$, $y M$, $M x y w_x w^{-1}$, $M x [1 + (b + 1) x] w_y w^{-1}$ plus terms $M w^{-1} x^i y^j$, $0 \leq i + j \leq 2$. Thus using (19) and (22) we get

$$(23) \quad \int \int_{H < h} (x^2 + y^2) M_x dx dy \\ = \frac{2}{b+1} \int \int_{H < h} M w^{-1} \left\{ (b+2) [x + (b+1)(x^2 - y^2)] \right. \\ \left. + \frac{c(b+3)}{1-b} [y + 2xy + 2cy^2] \right\} dx dy$$

$$(24) \quad \int \int_{H < h} (x^2 + y^2) M_y dx dy \\ = \frac{2}{b+1} \int \int_{H < h} M w^{-1} \{ c(x + (b+1)x^2) \\ + (b+2)(y + 2xy) + c(b+3)y^2 \} dx dy.$$

Now making use of (20) and (21) we eliminate in (23), (24) the terms containing x^2 , xy and y^2 . One obtains (with $\alpha = [(b^2 + c^2)(b+1)^2]^{-1}$)

$$\int \int_{H < h} (x^2 + y^2) M_x dx dy \\ = \alpha \int \int_{H < h} \{ [2(b^2 + c^2)(b+3) + 4b] M \\ + (b^2 + 2c^2 + b) M_x - c(b+1) M_y \} dx dy, \\ \int \int_{H < h} (x^2 + y^2) M_y dx dy \\ = \alpha \int \int_{H < h} \{ 4c M - c(b-1) M_x + b(b+1) M_y \} dx dy.$$

Finally, a straightforward computation yields that for the expression at λ_1 in M_1 one has $2M + xM_x + yM_y = -(2M + M_x)(b+1)^{-1}$.

Assume first that $\Delta = b^2 + c^2 - 1 > 0$. Denoting $X = 1 + (b+1)x + (c + \Delta^{1/2})y$, $Y = 1 + (b+1)x + (c - \Delta^{1/2})y$, we have

$$\begin{aligned} M_x &= -M(XY)^{-1} [(b+2)(X+Y) + c\Delta^{-1/2}(Y-X)], \\ M_y &= -M(XY)^{-1} [c(X+Y) + (\Delta+b-1)\Delta^{-1/2}(Y-X)]. \end{aligned}$$

After we take X, Y as the variables of integration in M_1 we get formula (iv)-(1).

Let $\Delta = 1 - b^2 - c^2 > 0$. Denoting $X = 1 + (b+1)x + cy$, $Y = y\Delta^{1/2}$, we have

$$\begin{aligned} M_x &= -2M(X^2 + Y^2)^{-1} [(b+2)X + c\Delta^{-1/2}Y], \\ M_y &= -2M(X^2 + Y^2)^{-1} [cX + (\Delta + 1 - b)\Delta^{-1/2}Y]. \end{aligned}$$

We take X, Y as new variables in the integral M_1 and obtain the formula in (iv)-(2).

In the case $b^2 + c^2 = 1$, we denote $X = 1 + (b+1)x + cy$, $Y = 2c(b+1)^{-1}y$. Then

$$\begin{aligned} M_x &= -MX^{-2} [2(b+2)X + (b+1)Y], \\ M_y &= -MX^{-2} [2cX + cY]. \end{aligned}$$

Integrating the equation $HX^{-3} = hX^{-3}$, one easily obtains the relation

$$\int \int_{H < h} MY X^{-2} dx dy = \frac{b-1}{b+1} \int \int_{H < h} M(X^{-2} - X^{-1}) dx dy$$

which yields the integral in (iv)-(3).

It remains to consider the case $b = -1$. In this case, $w = 1 + 2cy$, $M_x = -2M$ and $M_y = -(2c + 4y)Mw^{-1}$. Multiplying the equation $H = -\frac{1}{4}Mw(1+2x) = h$ by $x^j w^k$ and integrating over $H = h$, we get

$$\int \int_{H < h} Mx^j w^k [y + 2xy - 2kc(j+1)^{-1}x^2] dx dy = 0.$$

Using this relation with different j, k and also (19), we obtain via routine calculations that

$$M_1(h) = \frac{1}{c^2} \int \int M \{ \lambda_1 [(1 - c^2) w^{-1} - 1 - c^2] - \lambda_2 c^{-1} [(1 - c^2) w^{-1} + c^2 - w^2] + \lambda_3 [1 - w^2] \} dx dy.$$

We take $X = w$ and $Y = 2x + 2c^{-1}y$ as variables in the last integral to obtain (iv)-(4). Theorem 2 is proved.

4. Bifurcation functions for the degenerate cases

In this section we prove Theorem 3. The case (viii) related to the Hamiltonian triangle was already considered in [28], [10] and in this case the assertion is a consequence of results proved there. For the proof of (v)-(vii), we intend to use Corollary 1 at the end of section 2. According to that corollary, the computation of $M_2(h)$ can be reduced to the following procedure. The integrals in $M_1(h)$ which vanish on certain subset Γ of codimension one (or two) in the parameter space, are replaced with their derivatives with respect to the parameters. Afterwards the result is restricted onto Γ . The needed derivatives can be computed as follows. For definiteness, assume Γ is given by $c = 0$. With $H = H(x, y, c) = h$ and $I(h, c) = \int_{H=h} R(x, y, c) dx$, we have

$$(25) \quad \frac{d}{dc} I(h, c) = \int_{H=h} \left[R_c(x, y, c) - R_y(x, y, c) \frac{H_c}{H_y} \right] dx.$$

We need to use this general formula for several particular cases only. As in Theorem 2 above, the main part of the proof consists of long and straightforward calculations. Below we give them in more details for any of the cases. Recall that in the reversible cases $H = MX(\frac{1}{2}y^2 - U)$.

(v) Assume first that $b \neq -1$. Then by (18) and Corollary 1 we get $M_2(h) = [\lambda_1 I_1(h) + \lambda_3 I_3(h) + \lambda_2 \partial_a I_5(h)]|_{a=-1}$. Since I_5 contains a

factor $(a + 1)$, we have simply to divide by $a + 1$ and then to take a limit $a \rightarrow -1$. The result is $\partial_a I_5(h)|_{a=-1}$. Thus,

$$M_2(h) = -\frac{\lambda_1}{b+1} J_0 + \frac{\lambda_2}{3(b+1)^4} [2J_{-1} - (b+5)J_0 + (b+3)J_1] \\ - \frac{\lambda_3}{(b+1)^2} (J_0 - J_1)$$

which yields formula (v)-(1). When $b = -1$, we prefer to use the formula for the generic Hamiltonian case (i), that is $M_1(h) = 2 \int_{H=h} (\lambda_1 y + \lambda_2 y^2 + 2\lambda_3 xy) dx$. The integral at λ_2 , $I_2(h) = 2 \int_{H=h} y^2 dx$, vanishes for $c = 0$ and according to Corollary 1 we have to compute $\partial_c I_2|_{c=0}$. If $b = -1$, we have $H_y = y + c(x^2 - y^2)$, $H_c = x^2 y - \frac{1}{3} y^3$ (see point (i) of the appendix). Then formula (25) restricted on $c = 0$ yields

$$\frac{d}{dc} I_2|_{c=0} = 4 \int \int_{H < h} (x^2 - y^2) dx dy.$$

Expressing this integral in terms of J_0 and J_1 (we omit the details), we obtain for this case

$$M_2(h) = 2\lambda_1 J_0 + \frac{12}{11} \lambda_2 (2h J_0 - J_1) + 4\lambda_3 J_1$$

which proves Theorem 3 (v).

(vi) By (18) and Corollary 1 we have

$$M_2(h) = [\lambda_1 I_1(h) + \lambda_3 I_3(h) + \lambda_2 \partial_a I_5(h) + \lambda_4 \partial_b I_5(h)]|_{a=4, |b|=2}.$$

Denote for short $\mathcal{L} I_5 = (\lambda_2 \partial_a + \lambda_4 \partial_b) I_5|_{a=4, |b|=2}$. Provided $H = x^{\mu+1} [\frac{1}{2} y^2 - U(x, c)]$ and $R = M(x) x^k y = x^{\mu+k} y$ where $\mu = \mu(c)$, the general formula (25) reduces to

$$(26) \quad \frac{d}{dc} J_k(h, c) = \int_{H=h} M(x) x^k \left[\frac{d\mu}{dc} \left(\frac{y}{2} + \frac{U}{y} \right) \ln x + \frac{1}{y} \frac{dU}{dc} \right] dx.$$

From the proof of Theorem 1, we see that $I_5(h)$ vanishes for $a = 4$, $|b| = 2$ which is equivalent to $2J_{-1} - bJ_0 + (b-2)J_1 = 0$ (this identity can also be verified directly). We use it to obtain $\mathcal{L} I_5 = C[(\lambda_2 - \lambda_4)(J_0 - J_1) + \mathcal{L} J_*]$ where $C = -\frac{5}{32}$, $J_* = 2(J_{-1} - J_0)$

for $b = 2$ and $C = \frac{5}{1728}$, $J_* = 2(J_{-1} + J_0 - 2J_1)$ for $b = -2$. By (26) we further have

$$\mathcal{L} J_* = \int_{H=h} M(x) [2x^{-1} - b + (b-2)x] \left[\mathcal{L} \mu \left(\frac{y}{2} + \frac{U}{y} \right) \ln x + \frac{\mathcal{L} U}{y} \right] dx.$$

To obtain the need formula, we have to eliminate the terms in $\mathcal{L} J_*$ containing y^{-1} . This procedure is not too simple, however we will omit the details. They can be seen in the proof of statement (vii) below which we prefer to give in more details. The final expressions we obtain are

$$\begin{aligned} [\lambda_1 I_1 + \lambda_3 I_3] |_{a=4, |b|=2} &= \frac{1}{2(4-b)^2} \\ &\times \int \int M [(2-3b)\lambda_1 + 2\lambda_3 + (5(b-2)\lambda_1 - 2\lambda_3)x] dx dy, \end{aligned}$$

and, respectively for the cases $b = 2$ and $b = -2$, the integral $\mathcal{L} I_5$ equals

$$\begin{aligned} &\frac{5}{64} \int \int M [3\lambda_2 - 5\lambda_4 + 2(\lambda_4 - \lambda_2)x \\ &\quad + (3\lambda_4 - \lambda_2)x^{-2} + (3\lambda_2 - 5\lambda_4)(1 - x^{-1}) \ln x] dx dy, \\ &\frac{5}{10368} \int \int M (x-1) \times \\ &\quad \left[\lambda_2 - 35\lambda_4 - \frac{5}{2}(\lambda_2 + \lambda_4)x + (\lambda_2 + 5\lambda_4)(2 + x^{-1}) \ln x \right] dx dy. \end{aligned}$$

This yields the statement in (vi).

(vii) In real variables (x, y) the perturbed generic Lotka-Volterra system (see Theorem 1 (iv)) becomes, after rotation on angle $\pi/2$,

$$\begin{aligned} \dot{x} &= y + c(x^2 - y^2) + (2 - 2b)xy + \varepsilon[\lambda_1 x - \lambda_3(x^2 + y^2)], \\ \dot{y} &= -x - (1 + b)(x^2 - y^2) - 2cxy + \varepsilon[\lambda_1 y + \lambda_2(x^2 + y^2)]. \end{aligned}$$

We have $M_1(h) = \lambda_1 I_1(h) + \lambda_2 I_2(h) + \lambda_3 I_3(h)$ or explicitly, by (3),

$$\begin{aligned} M_1(h) &= \lambda_1 \int_{H=h} M(ydx - xdy) \\ &\quad + \lambda_2 \int_{H=h} M(x^2 + y^2) dx + \lambda_3 \int_{H=h} M(x^2 + y^2) dy. \end{aligned}$$

Clearly, the integral $I_2(h)$ vanishes at $c = 0$ (see the formulas in point (vii) of the appendix). Then for the reversible Lotka-Volterra system we obtain by Corollary 1: $M_2(h) = [\lambda_1 I_1(h) + \lambda_2 \partial_c I_2(h) + \lambda_3 I_3(h)]|_{c=0}$. Just as in the proof of (17) above, we derive the relation

$$(27) \quad \int \int_{H < h} M X^{-1} [(1 - 3b)y^2 + x + (b + 1)x^2] dx dy = 0$$

where $X = 1 + (2 - 2b)x$. In real coordinates, the perturbation functions for system (vii) in Theorem 1 (rotated) are respectively $f = x^2 - y^2 + \varepsilon [\lambda_1 x - \lambda_3 (x^2 + y^2)]$ and $g = -2xy + \lambda_2 (x^2 + y^2) + \varepsilon \lambda_1 y$. Then the fact that M_1 vanishes in combination with (27) yields

$$(28) \quad \begin{cases} M_1(h) = 4 \int \int_{H < h} M X^{-1} (x^2 - y^2) dx dy = 0, \\ \int \int_{H < h} M x dx dy = 0. \end{cases}$$

Now, via easy calculation one obtains

$$(29) \quad [\lambda_1 I_1 + \lambda_3 I_3]|_{c=0} = \begin{cases} (1 - b)^{-3} [\lambda_1 (1 - b) (J_{-1} - b J_0) + \lambda_3 (J_{-1} - J_0)], & b \neq 1, \\ 2 \lambda_1 J_0 + 8 \lambda_3 J_2, & b = 1. \end{cases}$$

The difficult part is the calculation of $\partial_c I_2(h)$. Write

$$(30) \quad \begin{cases} H(x, y, c) = H_0 + c H_*(x, y) + \dots, \\ M(x, y, c) = M_0 + c M_*(x, y) + \dots \end{cases}$$

where H_0 and M_0 coincide with H, M in point (vii) of the appendix. Using formula (25) we obtain

$$(31) \quad \partial_c I_2|_{c=0} = \int_{H_0=h} M_*(x^2 + y^2) dx - 2 \int_{H_0=h} \frac{H_*}{X} dx.$$

The expressions for H_* and M_* can be found explicitly since the general form of H and M is known: $H = l_1^{\alpha_1} l_2^{\alpha_2} l_3^{\alpha_3}$, $M = l_1^{\alpha_1-1} l_2^{\alpha_2-1} l_3^{\alpha_3-1}$, where α_j are constants and l_j are linear functions of x, y . Assume first

that $(b+1)(3b-1) > 0$ and denote for short $m = [(b+1)/(3b-1)]^{1/2}$, $n = [(b+1)(3b-1)]^{-1/2}$. Replacing (30) in (1) and taking into account the above remark we get that H_* and M_* are determined respectively by

$$\begin{aligned} H_* &= -\frac{2m^2 y H_0}{X} + \frac{2n H_0}{3b-1} \ln \left| \frac{y+mx+n}{y-mx-n} \right| \\ &\quad + \frac{y X M_0}{(3b-1)^2} \left[\frac{1-b}{1+b} + (2-4b)x \right], \\ M_* &= -\frac{4y M_0}{(3b-1)X} + \frac{2n M_0}{3b-1} \ln \left| \frac{y+mx+n}{y-mx-n} \right|. \end{aligned}$$

When $(b+1)(3b-1) < 0$, we have to replace $m = i[(b+1)/(1-3b)]^{1/2}$ and $n = i[(b+1)(1-3b)]^{-1/2}$ in the formulas above. For the rest of the proof, we will omit the subscript in H_0, M_0 thus keeping the previous notation. Let us consider first the general case when $b \neq \pm 1, 0, \frac{1}{3}$. Replacing the expressions of H_*, M_* in (31) and taking into account (28) we obtain

$$\begin{aligned} (32) \quad \partial_c I_2|_{c=0} &= \frac{J_{-1}}{(3b-1)(b-1)^3} - \frac{(b^2+6b-3)J_0}{(3b-1)^2(b-1)^3(b+1)} \\ &\quad + \frac{2n}{3b-1} \int_{H=h} M(2U+x^2) \ln \left| \frac{y+mx+n}{y-mx-n} \right| dx. \end{aligned}$$

We intend to prove below that the integral containing a logarithm (denote it by L) can be expressed as

$$\begin{aligned} (33) \quad L &= \frac{J_0}{b(3b-1)^2(b+1)} + \frac{1}{2b(3b-1)(b-1)(b+1)^2} \\ &\quad \times \int_{H=h} \frac{M(X-1)}{y} dx. \end{aligned}$$

Unfortunately, the proof of this fact requires rather long calculations. We first integrate by parts in L and use that, on the oval $H = \frac{1}{2}MX[y^2 - (mx+n)^2] = h$,

$$d \ln \left| \frac{y+mx+n}{y-mx-n} \right| = MX h^{-1} [my dx - (mx+n) dy]$$

to obtain

$$L = \frac{2b}{h(3b-1)^3(b-1)^3} \int_{H=h} M^2 y \left[X^4 - \frac{23b^3 + 7b^2 - 7b + 1}{4b^2(b+1)} X^3 \right. \\ \left. + \frac{(17b^3 + 9b^2 - b - 1)(3b-1)}{4b^2(b+1)^2} X^2 - \frac{(3b-1)^2}{(b+1)^2} X \right] dx.$$

Next, we denote $\mathcal{D} = 2hX^{(1+b)/(1-b)} + 2U$ and use that $\mathcal{D}' dx = 2y dy$ to get identity

$$\int \frac{M^2 X^j \mathcal{D}'}{y} dx = 4[4 + j(b-1)] \int M^2 X^{j-1} y dx \\ = 4[4 + j(b-1)] \int \frac{M^2 X^{j-1} \mathcal{D}}{y} dx.$$

Using it, we derive a further relation

$$(34) \quad 4(b-1)[4 + (j+1)(b-1)]L_{j-1} \\ = 4(b-1)(3b-1)hK_{j-2} + N_{j-1}, \quad j = 2, 3, \dots$$

where we have denoted, for short, $K_j = \int_{H=h} M X^j y^{-1} dx$, $L_j = \int_{H=h} M^2 X^j y dx$ and $N_j = \int_{H=h} M^2 X^j y^{-1} [(3b-1)(b+1)^{-1} - X] dx$. Operating with this basic recurrence formula we can express (after long reduction) integrals L_j , $1 \leq j \leq 4$, as linear combinations of integrals hK_j , $0 \leq j \leq 3$ and N_1 . We will not write up the explicit expressions. Replacing them in the formula of L above, we obtain

$$(35) \quad L = \frac{b}{(3b-1)^3(b-1)^3} \left(K_3 - \frac{11b^3 + 5b^2 - 5b + 1}{2b^2(b+1)} K_2 \right. \\ \left. + \frac{(3b-1)^2(3b^2 + 2b + 1)}{2b^2(b+1)^2} K_1 - \frac{(3b-1)^2}{(b+1)^2} K_0 \right)$$

(the terms containing N_1 annihilate). In a similar way we have derived (34), we get the relation

$$(36) \quad \int_{H=h} M X^{j-1} [x + (1+b)x^2] y^{-1} dx - [2j - 3 + (1-2j)b] \\ \times \int_{H=h} M X^{j-1} y dx = 0.$$

We use (36) to express K_2 and K_3 as linear combinations of K_0 , K_1 and J_0 (recall that $J_0 = J_1$ by (28)). Replacing the corresponding expressions in (35), we finally get (33). In combination, (33) and (32) yield

$$(37) \quad \partial_c I_2|_{c=0} = \frac{2}{b(b^2-1)(3b-1)} \\ \times \int_{H=h} M \left[\frac{X-1}{4(b+1)y} + y \left(\frac{3b-1}{b-1} - \frac{b^2+b}{(b-1)X} \right) \right] dx.$$

By (37) and formula (29) above we get the statement in Theorem 3 for the general case (vii)-(5).

Clearly formula (37) does not work if b coincides with any particular value $0, \frac{1}{3}, \pm 1$. For any of these values, we could repeat the procedure used for (37), which is however quite elaborate. For this reason, we proceed to obtain the needed formulas as a limit case of (37). We find this approach to be more useful. The direct limit in (37) as b tends to the particular values $0, \pm 1, \frac{1}{3}$ does not exist. To overcome this, we use the relations obtained above in order to remove firstly the terms which have a singular limit. Consider first the case $b = 1$. We add to (37) the integral $\int_{H=h} M xy dx = 0$ multiplied by a suitable constant and then take a limit $b \rightarrow 1$. One obtains

$$\partial_c I_2|_{c=0} = -\frac{1}{8} \int_{H=h} e^{-4x} [xy^{-1} + 4y(1+8x^2)] dx, \quad b = 1.$$

This formula combined with the formula (29) prove (vii)-(4). Next consider the case $b = 0$. The singular term in (37) in this case is $2b^{-1} \int_{H=h} M [\frac{1}{4}(X-1)y^{-1} + y] dx$. To remove this term we use the identity $M'_1(h) = -4 \int_{H=h} X^{-2}(x^2 - y^2)y^{-1} dx = 0$ written

as $\int_{H=h} M X^{\frac{2b}{1-b}} y^{-1} (x^2 - y^2) dx = 0$ and the identity (36) with $j = 1$. Multiply them by $1/b$ and $-1/b$ respectively and add to (37). Then for $b \rightarrow 0$ we obtain a limit $\partial_c I_2|_{c=0} = \int_{H=h} M [(2X^{-1} + 1)y + (x - x^2)y^{-1} + 2(x^2 - y^2)y^{-1} \ln X] dx$. Further, we apply (this time with $b = 0$) the same two identities to reduce this integral to $2 \int_{H=h} M [(X^{-1} - 1)y + (x^2 - y^2)y^{-1} \ln X] dx$. Finally, we make use of another relation $\int_{H=h} M \ln X [x + x^2] y^{-1} dx - \int_{H=h} M (2 - \ln X) y dx = 0$ (it is derived just as (36) and (34)) to simplify further the integral and get

$$\partial_c I_2|_{c=0} = \int_{H=h} M [(2X^{-1} - 6)y + (X^2 - X)y^{-1} \ln X] dx, \quad b = 0$$

which proves the case (vii)-(2). Let $b \rightarrow \frac{1}{3}$. The singular term in (37) can be written as $\frac{9}{4} (1 - 3b)^{-1} \int_{H=h} M (\frac{3}{4} xy^{-1} + 2yX^{-1}) dx$ and up to a constant multiplier it coincides with the left hand side of (36) taking there $b = \frac{1}{3}$, $j = 0$. Thus multiplying by a suitable constant (36) with $j = 0$ and adding to (37), we get after taking a limit $b \rightarrow \frac{1}{3}$ the expression $\partial_c I_2|_{c=0} = \frac{81}{32} \int_{H=h} M [(4 - 2X^{-1})y + (x + 2x^2)(Xy)^{-1}] dx$. To simplify the result, we use again (36) with $b = \frac{1}{3}$, $j = 0$, -1 and become to

$$\partial_c I_2|_{c=0} = \frac{81}{16} \int_{H=h} M (2 - 3X^{-1} + X^{-2}) y dx, \quad b = \frac{1}{3}.$$

To be precise, we note that $H(b)$ is not continuous at $b = \frac{1}{3}$. With $b > \frac{1}{3}$, we denote $h_0 = \frac{1}{8} (b + 1) (b - 1)^{-2} (3b - 1)^{-1}$ and consider $H(b) + h_0$ and the corresponding interval $(h_c + h_0, h_0)$ instead of $H(b)$ and $(h_c, 0)$. Then $H(b) + h_0 \rightarrow H(\frac{1}{3})$ and $(h_c + h_0, h_0)$ approaches the related interval $(\frac{9}{16}, \infty)$ as $b \rightarrow \frac{1}{3} + 0$ which makes correct the taking of limit. Finally, let $b \rightarrow -1$. Then there are two different singularities in (37). The leading singular term is $-\frac{1}{16} (b + 1)^{-2} \int_{H=h} M (X - 1) y^{-1} dx$ which can be removed via multiplying (36) with $j = 1$ by $\frac{1}{8} (1 - b) (b + 1)^{-2}$ and adding to (37). One obtains an expression which singularity is $\frac{1}{4} (b + 1)^{-1} \int_{H=h} M [(x^2 - x)y^{-1} - y] dx$. By (36) with $j = 1, 2$ we have the identity $\int_{H=h} M \{[x^2 + (1 + b)x^3]y^{-1} - y\} dx = 0$ which can then be used to get an integral possessing a limit as $b \rightarrow -1$. In this way we obtain the following expression $\partial_c I_2|_{c=0} = \frac{1}{8} \int_{H=h} M [(X^{-1} - 2)y - (2x^3 - x^2 + x)y^{-1}] dx$. As above, we have to add suitable constant

to H to make it continuous before to take a limit $b \rightarrow -1 - 0$. The last step consists in using (36) with $b = -1$ and $j = 1, 2, 3$ to remove the terms with y^{-1} . The final result is

$$\partial_c I_2|_{c=0} = \frac{1}{8} \int_{H=h} M(X^{-1} - X^2) y dx, \quad b = -1.$$

The case (vii) and together Theorem 3 is completely proved.

Remark 1. – The bifurcation function for the case (vii) derived in [28] contains an integral with a logarithm (the case considered there holds for values of b outside interval $[-1, \frac{1}{3}]$). The result was obtained by the following procedure [28, p. 251-252]. The succession function $I_{a,b} = \varepsilon^{-1} d(h, \varepsilon)$ is written as $I_{a,b} = I_1 + I_2$ where $I_1 \rightarrow I_{11} \neq 0$ and $I_2 \rightarrow 0$ for $\varepsilon \rightarrow 0$. Then the author finds the first variation of I_2 and takes $I_{11} + (d/d\varepsilon) I_2|_{\varepsilon=0}$ as the bifurcation function. As we have seen in the proof above, the integrals with logarithms can be removed (except for the case of isochronous center \mathcal{S}_1 given in (vii)-(2)).

5. Application to cyclicity problems

In this section we apply the formulas derived in Theorems 2 and 3 in studying the cyclicity of period annuli for several particular cases.

I. *The subspace \mathcal{A}_3 of centers in Q_3^R having (conjectured) cyclicity three period annuli.* – The bifurcation diagram of the component Q_3^R in the center manifold is more complicated because Q_3^R contains the line $a = 4$ of centers having cyclicity 3. The centers outside it have cyclicity 2. These facts suggest that the cyclicity of the period annulus will change by one when passing the line $a = 4$. It is believed that all the centers in Q_3^R having cyclicity three period annuli belong to the set \mathcal{A}_3 which is closed between the line $a = 4$ and the curve \mathcal{C}_3 of centers having cyclicity three separatrix contours. Therefore the curve \mathcal{C}_3 is an important element of the bifurcation diagram of Q_3^R . It is proved in [20] that \mathcal{C}_3 contains the half-line $a + b = 2$, $b \leq 0$ and goes through point Q_4^+ . Another observation in [20] is that the half-line $a - b = 2$, $b > 0$ lies in \mathcal{A}_3 . These facts were used in [20] to predict the possible location of the remaining (winding) part of \mathcal{C}_3 .

It is known that the cyclicity of loops and bounded segments in Q_3^R is two [25], [30]. As for the unbounded segments, the asymptotics derived in [31] imply that \mathcal{C}_3 is outside regions (II) and (VII), see Fig. 2.

Hence the winding part of \mathcal{C}_3 lies in the union of (III) and (IV). Denote $Q = \{(a, b) : a - b > 0, a + b - 2 > 0, a - 3b - 2 < 0\}$. For centers in Q , the period annulus is determined for $h \in (h_c, 0)$. The critical level $h_s = 0$ corresponds to the invariant hyperbola $\frac{1}{2}y^2 = U(x)$. The period annulus we consider is inside the right branch of the hyperbola. Given h , denote $\omega_1(h) = J_1(h)/J_0(h)$, $\omega_2(h) = J_{-1}(h)/J_0(h)$ and consider the curve in the (ω_1, ω_2) -plane

$$\Omega = \{(\omega_1(h), \omega_2(h)) : h \in (h_c, h_s)\}.$$

By Theorem 2 (ii)-(1), the number of zeros of $M_1(h)$ equals the number of intersection points of Ω with the line $\mu_1 + \mu_2 \omega_1 + \mu_3 \omega_2 = 0$ in the (ω_1, ω_2) -plane. Therefore the winding part of \mathcal{C}_3 is determined by the points $(a, b) \in Q$ at which the curve Ω changes the convexity near its endpoint $\Omega_0 = (\omega_1(0), \omega_2(0))$. In Lemmas 1 and 2 below we derive the asymptotic expansions needed for the proof of Theorem 4.

Remark 2. – It is well known that the ratio $\omega_1(h) = J_1(h)/J_0(h)$ is strictly monotone for perturbations of Q_3^R . Indeed, one can choose a perturbation for which the perturbed system (2) will have an invariant line and $M_1 = \mu_1 J_0 + \mu_2 J_1$. Then by the result of RYCHKOV (see COPPEL [4]), system (2) possesses at most one limit cycle which implies $\omega_1(h)$ is strictly monotone for $h \in (h_c, h_s)$. Hence Ω can be regarded as a graph of a function. In particular, $\omega_1(h)$ increases for $a > b$ and decreases for $a < b$. Moreover, by the Schwarz inequality, Ω lies inside the right branch of the hyperbola $\omega_1 \omega_2 = 1$.

LEMMA 1 (Asymptotic behaviour of Ω near $h = 0$ for centers in Q). – (i) The tangent of Ω at the point Ω_0 is $\omega_2 = \omega_2(0)$. (ii) For small negative h ,

$$\omega_2(h) - \omega_2(0) = \frac{W[J_0, J_{-1}](0)}{J_0^2(0)} h + o(h).$$

Proof. – We proceed to determine first the asymptotics of integrals $J_k(h)$, $k = 0, \pm 1$ near $h = 0$. Given a point $(a, b) \in Q$, write for short $H = x^{\mu+1}(\frac{1}{2}y^2 + Ax^2 + Bx + C)$ where $\mu < -3$, $A < 0$ and $B^2 - 4AC > 0$. Observing that $J'_k(h) = \int_{H=h} x^{k-1} y^{-1} dx$, we then have

$$J_k = -2^{3/2} \int_{x_1}^{x_2} x^{\mu+k} [\mathcal{D}(x, h)]^{1/2} dx,$$

$$J'_k = -2^{1/2} \int_{x_1}^{x_2} x^{k-1} [\mathcal{D}(x, h)]^{-1/2} dx$$

where $\mathcal{D}(x, h) = hx^{-\mu-1} - Ax^2 - Bx - C$ and $0 < x_1 < x_2$, $x_j = x_j(h)$ are the roots of $\mathcal{D} = 0$ determining the leftmost point $(x_1, 0)$ and the rightmost one $(x_2, 0)$ of the oval $\delta(h) \subset \{H = h\}$. Further, $x_1(h) \rightarrow x_1(0) > 0$, $x_2(h) \sim (h/A)^{1/(\mu+3)} \rightarrow \infty$ as $h \rightarrow -0$. All these formulas yield that $J_0, J_{-1} \in C^1[h_c, 0]$ and $J_1 \in C[h_c, 0]$, hence

$$J_k(h) = J_k(0) + J'_k(0)h + \dots, \quad k = -1, 0, \quad J_1(h) = J_1(0) + o(1).$$

To determine the second term in the asymptotic expansion of J_1 , we write $-J'_1$ as $2^{1/2}(\int_{x_1}^{\sqrt{x_2}} + \int_{\sqrt{x_2}}^{x_2}) = I_1 + I_2$. Consider first I_2 . Taking $s = xx_2$ as the new variable and using $\mathcal{D}(x_2, h) = 0$, we obtain after simple manipulations that

$$I_2 = x_2^{(\mu+3)/2} \left(-\frac{2}{h}\right)^{1/2} \int_{1/\sqrt{x_2}}^1 \left[\frac{1-s^{-\mu-3}}{1-s} + \varphi(s, h) \right]^{-1/2} \frac{ds}{\sqrt{s^2 - s^3}}$$

where $\varphi(s, h) = [Bx_2s + C(s+1)]x_2^{\mu+1}(hs^2)^{-1} = O(x_2^{-1/2})$ uniformly with respect to s on the range of integration. Thus, taking into account the asymptotics of x_2 , we get neglecting φ , that $I_2 \sim (-2A)^{-1/2} \ln x_2$. Similarly, we take $s = xx_1$ as the variable of integration to express I_1 in the form

$$I_1 = 2^{1/2} \int_1^{\sqrt{x_2}/x_1} \left[-A - \frac{B}{x_1(s+1)} + O(|h|^{1/2}) \right]^{-1/2} \times \frac{ds}{\sqrt{s^2 - 1}} \sim (-2A)^{-1/2} \ln x_2.$$

Therefore $J'_1(h) \sim c_1 \ln |h|$ and

$$J_1(h) = J_1(0) + c_1 h \ln |h| + \dots, \\ c_1 = \frac{4(a-b)^2}{\sqrt{(a+b-2)(3b-a+2)}}.$$

With the above asymptotics in hands, we get first (ii) and then

$$\frac{\omega_2(h) - \omega_2(0)}{\omega_1(h) - \omega_1(0)} \sim \frac{W[J_0, J_{-1}](0)}{c_1 J_0(0) \ln|h|}$$

which immediately yields (i). Lemma 1 is proved.

In the following lemma we find the asymptotic behaviour of Ω near the other endpoint at $h = h_c$. Note that, by the mean-value theorem, it is always $\Omega_1 = (\omega_1(h_c), \omega_2(h_c)) = (1, 1)$. We consider the general case $a \neq b$, $a \neq 3b + 2$ since the result below is also useful in studying the centers from Q_3^R outside Q .

LEMMA 2 (Asymptotic behaviour of Ω near $h = h_c$). – (i) *The tangent of Ω at the point Ω_1 is $\omega_2 - 1 = \frac{1}{2}(a - b - 2)(\omega_1 - 1)$.* (ii) *For small positive $h - h_c$,*

$$\begin{aligned} \omega_2(h) - 1 - \frac{1}{2}(a - b - 2)(\omega_1(h) - 1) \\ = \frac{8}{3}(a - b)^2(a - 4)(a - 3b - 2)(h - h_c)^2 + O(h - h_c)^3. \end{aligned}$$

Proof. – Instead of straightforward calculation of the asymptotics of J_k , we intend to use the formulas for $d(\xi, \varepsilon)$ and $M_1(h)$, see Sections 2, 3. Recall first that the Lyapunov quantities \bar{v}_j are rotational invariant [28]. Then, modulo an inessential factor (we will neglect it), $d_1(\xi) = (d/d\varepsilon)d(\xi, \varepsilon)|_{\varepsilon=0}$ coincides with $M_1(h)$ as derived in Section 3-(ii). Using the explicit expressions for \bar{v}_j , we obtain

$$\begin{aligned} d_1 = \lambda_1 \delta + \lambda_3 \left[4\pi\alpha + \frac{8}{3}\pi(2a - 3)b\beta + 10\pi(4 - b^2)b\gamma \right] \\ + \lambda_5 \left[\frac{8}{3}\pi(a^2 - 3a - 4)\beta + 10\pi(4 - b^2)(a + 1)\gamma \right]. \end{aligned}$$

Consider now the formula (18) for the bifurcation function $M_1(h)$. Comparing the coefficients at λ_j in d_1 and M_1 , we get

$$J_k = (a - b)[\delta + a_k\alpha + b_k\beta + c_k\gamma], \quad k = 0, \pm 1,$$

with

$$\begin{aligned}
 a_{-1} &= -2\pi(b+1)(a-b-2), \\
 b_{-1} &= \frac{4}{3}\pi(b+1)[a^2b - ab^2 + a^2 - 8ab + 9b^2 - 6a + 6b + 8], \\
 a_0 &= -2\pi(a+1)(a-b-2), \\
 b_0 &= \frac{4}{3}\pi(a+1)[a^2b - ab^2 + a^2 - 8ab + 9b^2 - 6a + 6b + 8], \\
 a_1 &= -2\pi(a^2 - ab - 3a + b - 2), \\
 b_1 &= \frac{4}{3}\pi[a^3b - a^2b^2 - a^3 - a^2b + 4ab^2 \\
 &\quad + a^2 - 14ab + 15b^2 + 10a - 2b + 8].
 \end{aligned}$$

Thus we find, taking into account the asymptotics of δ , α , β , γ , that

$$\omega_2(h) - 1 - \frac{a_{-1} - a_0}{a_1 - a_0} (\omega_1(h) - 1) = \frac{\beta}{\delta} \left[b_{-1} - b_0 - \frac{a_{-1} - a_0}{a_1 - a_0} (b_1 - b_0) \right] + \dots$$

Using that $h - h_c \sim \frac{1}{2}\xi^2$ for ξ small and the above expressions, we get the statement of the lemma.

Proof of Theorem 4. – By the asymptotics derived in Lemma 1,

$$M_1(h) = \mu_1 J_0(h) + \mu_2 J_1(h) + \mu_3 J_{-1}(h) = m_0 + m_1 h \ln|h| + m_2 h + \dots$$

Then $m_0 = m_1 = m_2 = 0$ yields $\mu_1 = \mu_2 = \mu_3 = 0$ except for the points $(a, b) \in Q$ satisfying $W[J_0, J_{-1}](0) = 0$. This proves the assertion that the winding part of the curve \mathcal{C}_3 (of points in the parameter space corresponding to systems with cyclicity three separatrix contours) has an equation $W[J_0, J_{-1}](0) = 0$, provided $b > 0$.

Using now Lemmas 1 and 2, we immediately obtain that in $Q \cap \mathcal{A}_3$, the ends of the curve Ω have different convexity. This yields that in \mathcal{A}_3 the cyclicity of the period annulus is at least 3 (outside Q we apply the result from [20]). In $Q \setminus \mathcal{A}_3$, the ends of Ω have the same convexity. This fact indeed does not suffice to conclude rightly that Ω is convex (concave) outside \mathcal{A}_3 which would guarantee cyclicity 2 of annuli there.

As in the proof of Lemma 1, denote $\mu = -(2a+2)/(a-b)$ and let $\nu = x_0/x_1$ where $x_0 < x_1$ be the roots of $U(x) = 0$. Then expressing

$W[J_0, J_{-1}](0)$ as a double integral, we can write the equation of the winding line $W = 0$ in the form

$$(38) \quad \Phi(\mu, \nu) = \int_0^1 z^{-\mu-3} (z - \varphi(\nu)) [(1-z)(1-\nu z)]^{1/2} dz = 0,$$

$$\mu < -3, \quad \nu < 1$$

where the function $\varphi(\nu) = (\nu+1)/(2\nu) - (\nu \int_0^1 [(1-s)(1-\nu s)]^{-1/2} ds)^{-1}$ is elementary. For $\nu \rightarrow -\infty$, we have $\varphi(\nu) = \frac{1}{2} + \pi^{-1}(-\nu)^{-1/2} + \dots$. Then routine asymptotic calculations (here and below the details are omitted) yield $\mu(\nu) = -3 - 6\pi^{-1}(-\nu)^{-1/2} + \dots$ for the solution of (38). This determines the asymptotic expansion

$$b = 2 - a + \frac{16}{9} \pi^2 (2-a)^3 + \dots$$

of the winding curve near the isochronous center S_2 . Similarly, near $\nu = 0$ we have $\varphi(\nu) = \frac{2}{3} + \frac{2}{45}\nu + \dots$ and the corresponding solution of (38) is $\mu(\nu) = -5 - \frac{48}{55}\nu + O(\nu^2)$. This yields the asymptotics of C_3 near Q_4^+ :

$$b = 2 + \frac{69}{97}(a-4) + O((a-4)^2).$$

Moreover, the equation $\Phi(-5, \nu) = 0$ has a unique solution $\nu = 0$. This fact can be verified numerically since $\Phi(-5, \nu)$ is an elementary function. Notice that equation $\mu = -5$ defines the line $\ell_4 : 3a - 5b - 2 = 0$ going through Q_4^+ and the isochronous center $S_4 = (\frac{7}{3}, 1)$. Hence, ℓ_4 lies outside \mathcal{A}_3 . On the other hand, the line $\ell_2 : a - b - 2 = 0$ passing through Q_4^+ and S_2 , lies for $b > 0$ inside \mathcal{A}_3 [20] (this also follows directly from the fact that, by Lemmas 1 and 2, on ℓ_2 the slope of tangents to both endpoints of Ω is zero). We have thus obtained: *the winding part of C_3 is tangential to $a + b - 2 = 0$ and is entirely placed inside the acute cone formed by ℓ_2 and ℓ_4 (see Fig. 4).*

Let us also consider in brief the case when there are two centers in Q_3^R (see Fig. 2). This situation occurs if (a, b) is in domain $C_2 : \{(a+b-2)(a-3b+2) > 0\}$. The involution \mathcal{I} interchanging the centers acts in $C_2 \setminus \{b = 0\}$ and is given analytically by $\mathcal{I}(a, b) = ((a-b+1)/b, 1/b) = (a', b')$. Under the action of \mathcal{I} , the lines through $(a, b) = (-1, -1)$ are stationary and $b = 1$ consists of stationary points.

If \mathcal{A}'_3 is the image of the part of \mathcal{A}_3 contained in $C_2 \setminus \{b = 0\}$, then the points $(a, b) \in \mathcal{A}'_3$ present, in the bifurcation diagram of Q_3^R , the systems having cyclicity three period annulus around the second center $z \neq 0$, see Figure 5. In particular, the intersection $\mathcal{A}_3 \cap \mathcal{A}'_3$ presents systems with cyclicity three period annuli around both centers. Let us remark that when $b = 0$, the center obtained after interchanging is degenerate and falls into the reversible part of Q_3^{LV} .

II. *The reversible codimension four center $Q_4^+ = Q_3^R \cap Q_4$, $b = 2$.* – In this paragraph we investigate the integral $M_2(h)$ giving the expression for the bifurcation function in the case (vi)-(1) of Theorem 3. Our approach for estimating the number of zeros of M_2 is based on the possibility to derive appropriate Fuchs equations satisfied by the integrals included

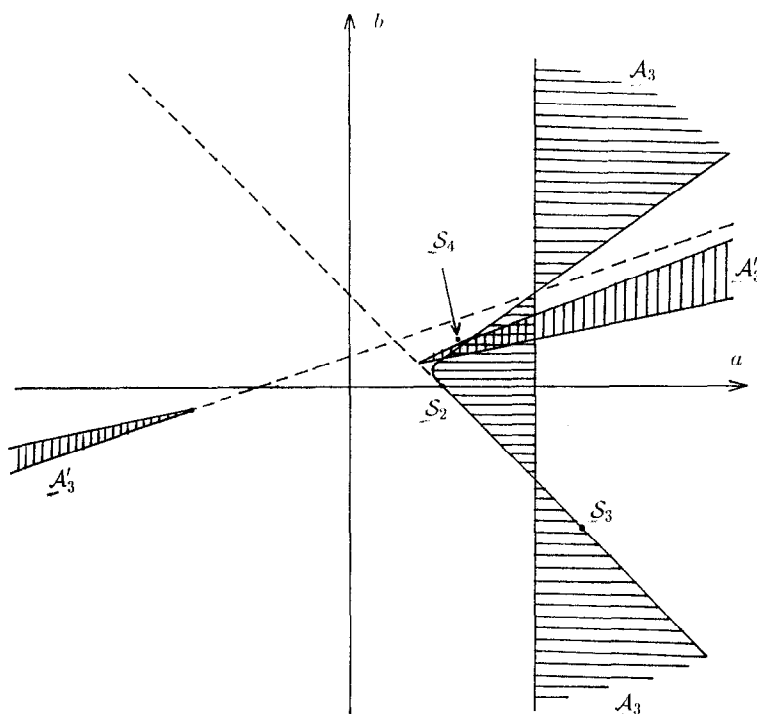


Fig. 5. – Location of domains \mathcal{A}_3 and \mathcal{A}'_3 in the bifurcation diagram of Q_3^R in the (a, b) -plane.

The horizontally shaded area corresponds to systems for which the center at the origin has cyclicity three period annulus. The vertically shaded area corresponds to systems for which the second center (outside the origin) has cyclicity three period annulus.

in M_2 . Performing a suitable scaling of y , H and h , we can assume (see the appendix below) that $H = x^{-4}(y^2 - x^2 + \frac{2}{3}x) = h$. In these coordinates, the ovals are defined for $h \in (-\frac{1}{3}, 0)$. With $M = x^{-5}$ and $y^2 = \mathcal{D}(x, h) = hx^4 + x^2 - \frac{2}{3}x$ we have

$$\begin{aligned} \int_{H=h} M y x^k \mathcal{D}' dx &= \frac{2}{3} \int_{H=h} M x^k dy^3 \\ &= -\frac{2}{3}(k-5) \int_{H=h} M y x^{k-1} \mathcal{D} dx \end{aligned}$$

and similarly

$$\begin{aligned} \frac{1}{2} \int_{H=h} M x^{k+4} \mathcal{D}' y^{-1} dx &= -(k-1) J_{k+3}, \\ \frac{1}{2} \int_{H=h} M x^{k+3} \mathcal{D} y^{-1} dx &= \frac{1}{2} J_{k+3}. \end{aligned}$$

Recall that $J_k(h) = \int_{H=h} M x^k y dx$. Then $J'_k(h) = \frac{1}{2} \int_{H=h} M x^{k+4} y^{-1} dx$ and the identities above yield

$$\begin{aligned} (k+1)h J_{k+3} + (k-2) J_{k+1} + \frac{1}{3}(7-2k) J_k &= 0, \\ (39) \quad 4h J'_{k+3} + 2 J'_{k+1} - \frac{2}{3} J'_k + (k-1) J_{k+3} &= 0, \\ h J'_{k+3} + J'_{k+1} - \frac{2}{3} J'_k - \frac{1}{2} J_{k+3} &= 0. \end{aligned}$$

We point out that the above relations also hold for any real k . In particular, we have

$$(40) \quad J_{-1} = J_0, \quad J_{-2} = \frac{12}{11} J_0 + \frac{3}{11} h J_1.$$

Denoting for short $J_* = \int_{H=h} M y (1-x^{-1}) \ln x dx$, we therefore obtain by Theorem 3 the following expression for the second Melnikov function:

$$M_2(h) = \left(\mu_1 + \frac{12}{11} \mu_3 \right) J_0(h) + \left(\mu_2 + \frac{3}{11} h \mu_3 \right) J_1(h) + \mu_4 J_*(h).$$

Our first goal will be to derive a three-dimensional Fuchsian system satisfied by the integrals J_0 , J_1 and J_* . We follow the idea from [10]. We

begin by the observation that, by the first equation in (40), the non-Abelian integral J_* can be considered as the limit $J_* = \lim_{\omega \rightarrow 0} (J_\omega - J_{\omega-1})/\omega$ (clearly, $J_{k+\omega} \rightarrow J_k$ for $\omega \rightarrow 0$, k integer). The second and third equations in (39) yield $3h J'_{k+3} + J'_{k+1} + (k - \frac{1}{2}) J_{k+3} = 0$. We next replace in this equation $k = \omega - 2, \omega - 3, \omega - 4$ and then, with the help of the first equation in (39), eliminate from the obtained system the integrals $J'_{\omega-3}$ and $J'_{\omega-2}$. The result is

$$\begin{aligned} 3h J'_{\omega+1} + J'_{\omega-1} + \left(\omega - \frac{5}{2}\right) J_{\omega+1} &= 0, \\ (\omega - 1)h J'_{\omega+1} + (2\omega - 11)h J'_\omega + (\omega - 4)J'_{\omega-1} \\ + (\omega - 1)J_{\omega+1} + \left(\frac{2}{3}\omega^2 - 6\omega + \frac{77}{6}\right)J_\omega &= 0, \\ (13 - 2\omega)h J'_\omega + (2\omega - 13)h J'_{\omega-1} \\ - \left(\omega^2 - \frac{19}{2}\omega + \frac{39}{2}\right)J_\omega + \left(\frac{2}{3}\omega^2 - \frac{22}{3}\omega + \frac{39}{2}\right)J_{\omega-1} &= 0. \end{aligned}$$

We put $J_{\omega-1} = J_\omega - \omega D_*$ in the third equation of above system and then divide the result by 13ω . Afterwards take a limit $\omega \rightarrow 0$ in the resulting system. One obtains

$$\begin{aligned} 3h J'_1 + J'_0 - \frac{5}{2}J_1 &= 0 \\ -h J'_1 - (4 + 11h)J'_0 - J_1 + \frac{77}{6}J_0 &= 0 \\ h J'_* - \frac{3}{2}J_* + \frac{1}{6}J_0 &= 0. \end{aligned}$$

Solving with respect to $\mathbf{J} = \text{col}(J_0, J_1, J_*)$, we get the needed system in the form

$$(41) \quad \mathbf{J} = \mathbf{A}(h) \mathbf{J}', \quad \text{where } \mathbf{A}(h) = \begin{pmatrix} 6h/7 + 12/35 & 6h/35 & 0 \\ 2/5 & 6h/5 & 0 \\ 2h/21 + 4/105 & 2h/105 & 2h/3 \end{pmatrix}.$$

As is well known, the most of the sharp results in perturbations (not necessarily concerning quadratic centers) were obtained in situations where the corresponding bifurcation function takes the form $h \rightarrow P_1(h)I_1 + P_2(h)I_2$, in which P_k are polynomials and the ratio of Abelian integrals

I_1/I_2 satisfies certain Riccati equation [11], [13], [15-19], [26]. However, very few such examples are known. They arise when the unperturbed field $dH = 0$ is Hamiltonian and perhaps include only $H = y^2 \pm x^4 + ax^2$ and the Bogdanov-Takens Hamiltonian $H = y^2 + x^2 - x^3$. For instance, except for the Bogdanov-Takens case, all other cubic Hamiltonians lead to first order Picard-Fuchs systems of size three (or even four) for the integrals appearing in the bifurcation function. Thus in general it is impossible to use them in deriving the required Riccati equation. Therefore a different approach is needed to all these cases. As mentioned in [6] where a subset in the space of reversible cubic Hamiltonians was considered, the second derivative $M_1''(h)$ should satisfy a second order Picard-Fuchs equation. This fact can be explained by using algebraic geometry (see [8]) and remains also true for all other cubic Hamiltonians. It turns out that in some situations [8], [10] it is possible to develop this idea and using the related Riccati equation to find a sharp estimate for the number of limit cycles. Analytically, the possibility to derive in the Hamiltonian case a Riccati equation for M_1'' from the Picard-Fuchs system relies upon the fact that the latter has the form $\mathbf{J} = \mathbf{A}(h)\mathbf{J}'$, where the constant matrix \mathbf{A}' possesses an eigenvalue equal to 1. Thus after we differentiate, one obtains $(\mathbf{I} - \mathbf{A}')\mathbf{J}' = \mathbf{A}(h)\mathbf{J}''$ and the matrix $\mathbf{I} - \mathbf{A}'$ is singular. This implies that the components of \mathbf{J}'' will satisfy a linear equality which can be used to obtain a Riccati equation for the ratio of two of the components.

The situation in the non-Hamiltonian case is quite different, even for those few cases where it is still possible to derive a Fuchsian system for the entries of the bifurcation function. The matrix \mathbf{A}' has no longer 1 as the eigenvalue and the approach used for the Hamiltonian case does not work elsewhere.

In the considered case, we can proceed as follows. Rewrite for short the bifurcation function as $M_2(h) = \alpha J_0(h) + (\beta h + \gamma) J_1(h) + \delta J_*(h)$. By (41) we get that the ratio $w = J_0/J_1$ satisfies the Riccati equation

$$\begin{aligned}
 6h(3h+1)w' &= 7w^2 + 6(h-1)w - 3h, \\
 (42) \quad \text{or} \quad \dot{h} &= 6h(3h+1), \\
 \dot{w} &= 7w^2 + 6(h-1)w - 3h.
 \end{aligned}$$

Next, a straightforward calculation using (41) yields the following linear equation for $M_2(h)$:

$$(43) \quad 6h M_2' - 9M_2 = \frac{P_2(h) J_1 - P_1(h) J_0}{3h + 1},$$

where

$$\begin{aligned} P_2(h) &= 6\beta h^2 - (3\alpha - 3\beta + 12\gamma)h - 3\gamma, \\ P_1(h) &= (6\alpha + 7\beta + 3\delta)h + (9\alpha + 7\gamma + \delta). \end{aligned}$$

Solving the above equation with respect to M_2 , we get the representation

$$(44) \quad \begin{aligned} M_2(h) &= \frac{|h|^{3/2}}{6} \int_{-\frac{1}{3}}^h \frac{[P_1(z) J_0(z) - P_2(z) J_1(z)] dz}{|z|^{5/2} (3z + 1)}, \\ h &\in \left(-\frac{1}{3}, 0\right) \end{aligned}$$

(we recall that J_k, J_* and hence M_2 vanish at $h = -\frac{1}{3}$). Clearly the number of zeros of M_2 in $(-\frac{1}{3}, 0)$ does not exceed the number of zeros of the numerator $N(h) = P_1(h) J_0(h) - P_2(h) J_1(h)$ in the same interval (in fact, the zeros of $N(h)$ are just the critical points of the function defined by the integral in (44)). Therefore the problem reduces to estimating the zeros of $N(h)$ in $(-\frac{1}{3}, 0)$. For this purpose, we can use the equation (42). Simple observations show (see below) that $N(h)$ can have at most three zeros. This proves Theorem 5.

Clearly, a similar approach is also applicable in more general situations, where the bifurcation function is of form $R_1(h) I_1 + R_2(h) I_2 + R_3(h) I_3$, with R_k rational, and the corresponding first order Fuchsian system satisfied by $\mathbf{I} = \text{col}(I_1, I_2, I_3)$ includes two dimensional invariant subsystem.

To investigate the possible zeros of $N(h)$, we consider the phase portrait of system (42), see Figure 6. The system has two saddle points $S_0(0, 0)$, $S_1(-\frac{1}{3}, 1)$ and two improper nodes $N_0(-\frac{1}{3}, \frac{1}{7})$, $N_1(0, \frac{6}{7})$. The graph Γ of $w(h)$ coincides with the stable separatrix of the saddle point S_1 connecting S_1 and N_1 . It is easily seen that $w(h)$ is strictly decreasing and concave for $h \in (-\frac{1}{3}, 0)$. Clearly, the number of zeros of

$N(h)$ equals the number n of intersections between Γ and the hyperbola $d = d(h) = P_2(h)/P_1(h)$. We have $d(-\frac{1}{3}) = \frac{1}{7}$ and hence the hyperbola goes through N_0 . Then using elementary geometric observations we conclude that $n \leq 2$ for any of the cases 1) $d'(h) \neq 0$, that is d is monotone; 2) $d'(h)$ has two zeros and Γ does not intersect the lower (concave) branch of the hyperbola. If Γ has a common point with the lower branch (denote it by d_-), then clearly $N_0 \in d_-$ and Γ cannot intersect the upper branch too. Denote by Q_1 the first point in $d_- \cap \Gamma$, starting from N_0 . The phase portrait implies that d_- contains between N_0 and Q_1 a contact point C_0 with the flow in (42) (see Fig. 7). The equation of the contact points $(d/dt)[w(t) - d(h)]|_{(42)} = \dot{w} - \dot{h}d'(h)|_{w=d} = 0$ has the form $(3h + 1)P_3(h)P_1^{-2}(h) = 0$ where P_3 is a cubic polynomial. Thus there are at most two other contact points C_1, C_2 which gives $n \leq 3$. The exceptional cases when d reduces to parabolic or linear function are elementary as well.

III. *The isochronous center \mathcal{S}_1 .* – In [3] the authors investigated the number of the limit cycles bifurcating in small quadratic perturbations of quadratic systems with an isochronous center. They found that the cyclicity of the corresponding annulus is 3 for the (linear) isochrone \mathcal{S}_0 , 1 for the isochronous center \mathcal{S}_1 and 2 for the other isochrones $\mathcal{S}_2, \mathcal{S}_3$ and \mathcal{S}_4 . However, the result about \mathcal{S}_1 is not correct. Below we prove that the cyclicity of the annulus around \mathcal{S}_1 also is 2. The assertion of Theorem 4.6 in [3] does not suffice to determine the cyclicity of the annulus around \mathcal{S}_1 as assumed there. The maximum of limit cycles for this case is attained if $\lambda_5|_{\varepsilon=0} = 0$ in the Bautin normal form. This possibility however is not considered in [3, §4]. Thus in fact the analysis concerning \mathcal{S}_1 in [3] is only first order in ε , whilst \mathcal{S}_1 is the unique quadratic isochrone requiring a second order analysis. We remark that the isochronous centers $\mathcal{S}_k \in Q_3^R$, $k = 2, 3, 4$ are given by values $(a, b) = (2, 0), (5, -3)$ and $(\frac{7}{3}, 1)$ respectively (see [14]) and hence they fall into the generic cases. The isochrone \mathcal{S}_1 is given by $(b, c) = (0, 0)$ in Q_3^{LV} and hence falls within the degenerate cases.

According to Theorem 3, the related bifurcation function is given by formula (vii)-(2). To simplify the calculations, we additionally perform a scaling of y, H and h (which does not affect the formula of M_2) to obtain the first integral into the form $H = x^{-1}(y^2 + x^2 + 1) = h$. The period annulus around the center $(1, 0)$ is determined for $h \in (2, \infty)$ where $h = 2$

is the critical level corresponding to the isochronous center itself. Denote for short $J_*(h) = \int_{H=h} M y (x-1) \ln x \, dx$ ($M = x^{-2}$ in the considered case). Then $M_2(h) = \mu_1 J_0(h) + \mu_2 J_{-1}(h) + \mu_3 J'_*(h)$. As far as the level curves are circles, the integrals can be computed explicitly. However the direct calculation is long and for this reason we will obtain the needed formulas by using the Fuchsian equations satisfied by J_k, J_* . Just as in the previous paragraph, we derive the relations

$$(2k-1)hJ_k - 2(k+1)J_{k+1} - 2(k-2)J_{k-1} = 0,$$

$$hJ'_{k+1} - J'_k - J'_{k+2} = \frac{1}{2}J_{k+1}, \quad hJ'_k - 2J'_{k+1} = -(k-1)J_k.$$

Further, we have $J_{k+1} = kD_* + J_k$ where $D_* \rightarrow J_*$ as $k \rightarrow 0$. Using this fact and the above formulas, we get after a simple manipulation the

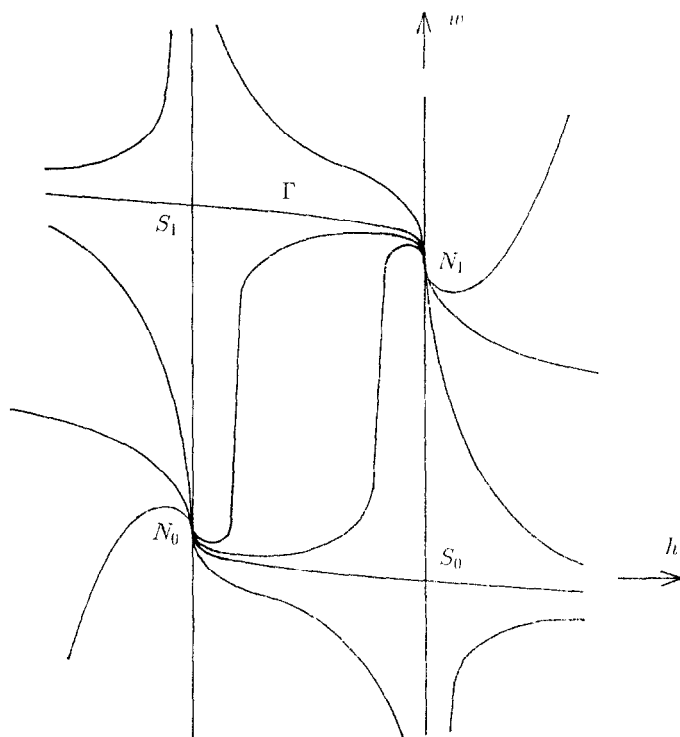


Fig. 6. – Phase portrait of system (42).

equations $(h-2)J'_0 = J_0$, $(h+2)J'_* = J_* + 2J_0$. Since all integrals vanish for $h=2$ and $J_2(h) = \pi(1 - \frac{1}{4}h^2) = -(\text{the area bounded by the oval } H=h)$, all these equations yield

$$J_0 = \pi(2-h), \quad J_{-1} = \pi\left(1 - \frac{1}{4}h^2\right),$$

$$J_* = 2\pi\left[h-2 - (h+2)\ln\frac{1}{4}(h+2)\right].$$

Hence

$$M_2(h) = \pi\left[\mu_1(2-h) + \mu_2\left(1 - \frac{1}{4}h^2\right) - 2\mu_3\ln\frac{1}{4}(h+2)\right],$$

$$M'_2(h) = \pi\left[-\mu_1 - \frac{1}{2}\mu_2 h - 2\mu_3(h+2)^{-1}\right].$$

As the derivative M'_2 has at most two zeros at all, the bifurcation function M_2 has no more than two zeros in $(2, \infty)$ (the third zero is

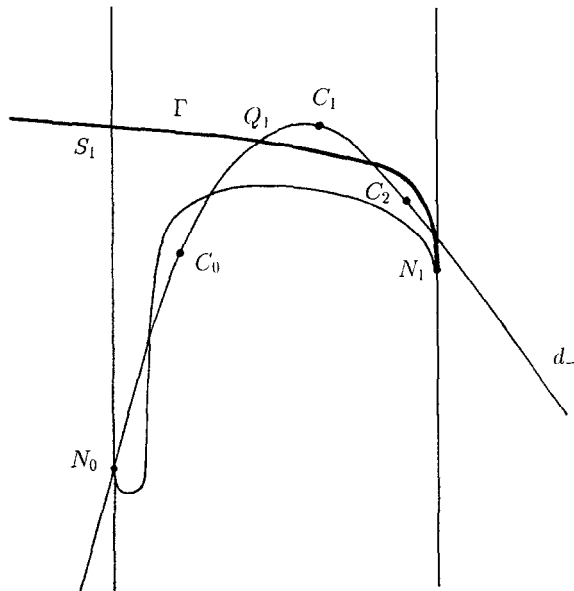


Fig. 7. — Contact points of the flow in (42) with the hyperbola $d(h) = P_2(h)/P_1(h)$.

always $h = 2$). Thus we conclude that the cyclicity of the period annulus is at most two. It is an easy task to construct a perturbation with two cycles. For example, we can choose μ_j so that $\mu_2 < 0$, $M_2(6) < 0$, $M'_2(2) > 0$, e.g. $\mu = (\frac{9}{2}, -\frac{5}{6}, -8)$. Then $M_2(h)$ will have two zeros $h_1 \in (2, 6)$ and $h_2 \in (6, \infty)$.

6. Appendix

Below we give the list of all first integrals of the quadratic integrable systems. Although we write up the original equations, all listed first integrals correspond to coordinates obtained from the original ones by a rotation on an angle $\pi/2$ (which results in replacing (x, y) by $(y, -x)$). This is done because we prefer to have in the reversible cases an axial symmetry with respect to the abscissa $y = 0$. In all cases where reasonable, we express H directly through the original variables (rotated) x, y and parameters a, b, c avoiding the introduction of new symbols. The arguments of H, M for each case are chosen to coincide with the variables of integration used in Theorem 2 and 3.

(i) *The Hamiltonian system* $Q_3^H [\dot{z} = -iz - z^2 + 2|z|^2 + (b + ic)\bar{z}^2]$

$$H(x, y) = \frac{1}{2}(x^2 + y^2) + \left(\frac{1}{3}b - 1\right)x^3 + cx^2y - (1 + b)xy^2 - \frac{1}{3}cy^3.$$

(ii) *The reversible system* $Q_3^R \setminus Q_3^H [\dot{z} = -iz + az^2 + 2|z|^2 + b\bar{z}^2, a \neq -1]$

(1a) The general case: $X = 1 + 2(a - b)x$, $M(X) = X^{-\frac{2a+2}{a-b}}$,

$$H(X, y) = X^{-\frac{a+b+2}{a-b}} \left(\frac{y^2}{2} + \frac{1}{8(a-b)^2} \times \left(\frac{a+b-2}{a-3b-2} X^2 + 2 \frac{b-1}{b+1} X + \frac{a-3b+2}{a+b+2} \right) \right).$$

(1b) Case $a + b + 2 = 0$: $X = 1 + 4(a + 1)x$,

$$H(X, y) = \frac{1}{2}y^2 - \frac{1}{16}(a + 1)^{-3} \left[\frac{1}{2}X^2 - (a + 3)X + (a + 2)\ln X \right],$$

$$M(X) = X^{-1}.$$

(1c) Case $b = -1$: $X = 1 + 2(a+1)x$.

$$H(X, y) = X^{-1} \left(\frac{1}{2} y^2 + \frac{1}{8} (a+1)^{-3} [(a-3)X^2 + 8X \ln X + a+5] \right),$$

$$M(X) = X^{-2}$$

(2) Case $a = 3b + 2$: $X = 1 + \frac{4}{3}(a+1)x$,

$$H(X, y) = X^{-2} \left(\frac{1}{2} y^2 + \frac{9}{16} (a+1)^{-3} \left[(a-2)X^2 \ln X + (a-5)X + \frac{3}{2} \right] \right),$$

$$M(X) = X^{-3}.$$

(3) Case $a = b$ (invariant line at infinity):

$$H(x, y) = e^{-4(a+1)x} \left(\frac{y^2}{2} - \frac{a-1}{2(a+1)} x^2 - \frac{a}{2(a+1)^2} x - \frac{a}{8(a+1)^3} \right),$$

$$M(x) = e^{-4(a+1)x}.$$

(iii) *The codimension four system $Q_4 \setminus Q_3^R$*

$$[\dot{z} = -iz + 4z^2 + 2|z|^2 + (b+ic)\bar{z}^2, c \neq 0, |b+ic| = 2] :$$

$$X = 1 + 8x + 4(2+b)x^2 + 8cxy + 4(2-b)y^2, Y = (2+b)x + cy,$$

$$H(X, Y) = \frac{X^{-3/2}}{8(2-b)} \left(\frac{4}{3(2+b)} Y^3 + \frac{4}{2+b} Y^2 + (1-X)Y - X + \frac{1}{3} \right),$$

$$M(X) = X^{-5/2}.$$

(iv) *The generic Lotka-Volterra system Q_3^{LV}*

$$[\dot{z} = -iz + [1 + \frac{1}{2}(b-ic)]z^2 + \frac{1}{2}(b+ic)\bar{z}^2]$$

$$(1) \text{ Case } \Delta = b^2 + c^2 - 1 > 0 : X = 1 + (b+1)x + (c + \Delta^{1/2})y,$$

$$Y = 1 + (b+1)x + (c - \Delta^{1/2})y,$$

$$H(X, Y) = \frac{1}{2} (b-1)^{-1} (1+2x) X^\lambda Y^\mu,$$

$$M(X, Y) = X^{\lambda-1} Y^{\mu-1},$$

$$\text{where } \lambda = (c\Delta^{-1/2} - 1)(b+1)^{-1}, \mu = (-c\Delta^{-1/2} - 1)(b+1)^{-1}.$$

(2) Case $\Delta = 1 - b^2 - c^2 > 0$: $X = 1 + (b + 1)x + cy$, $Y = \Delta^{1/2}y$,

$$H(X, Y) = \frac{1}{2}(b - 1)^{-1}(1 + 2x)(X^2 + Y^2)^\lambda e^{\mu \operatorname{Arctan}(Y/X)},$$

$$M = (X^2 + Y^2)^{\lambda-1} e^{\mu \operatorname{Arctan}(Y/X)},$$

where $\lambda = -(b + 1)^{-1}$, $\mu = 2c(b + 1)^{-1} \Delta^{-1/2}$.

(3) Case $b^2 + c^2 = 1$ (double invariant line): $X = 1 + (b + 1)x + cy$,
 $Y = 2c(b + 1)^{-1}y$,

$$H(X, Y) = \frac{1}{2}(b - 1)^{-1}(1 + 2x)X^{2\lambda}e^{Y/X},$$

$$M(X, Y) = X^{2\lambda-2}e^{Y/X}.$$

(4) Case $b = -1$ (invariant line at infinity): $X = 1 + 2cy$, $Y = 2x + 2c^{-1}y$

$$H(X, Y) = -\frac{1}{4}(1 + 2x)X^{1/c^2}e^{-Y}, \quad M(X, Y) = X^{1/c^2-1}e^{-Y}.$$

(v) *The reversible Hamiltonian system $Q_3^H \cap Q_3^R$*
 $[\dot{z} = -iz - z^2 + 2|z|^2 + b\bar{z}^2]$

(1) Case $b \neq -1$: $X = 1 - 2(b + 1)x$,

$$H(X, y) = X \left(\frac{1}{2}y^2 + \frac{1}{8}(b + 1)^{-3} \left[\left(1 - \frac{1}{3}b \right) X^2 + 2(b - 1)X + 1 - 3b \right] \right)$$

(2) Case $b = -1$ (Bogdanov-Takens Hamiltonian):

$$H(x, y) = \frac{1}{2}(x^2 + y^2) - \frac{4}{3}x^3.$$

(vi) *The reversible codimension four system $Q_4 \cap Q_3^R$*
 $[\dot{z} = -iz + 4z^2 + 2|z|^2 \pm 2\bar{z}^2]$

(1) Case $+$: $X = 1 + 4x$, $H(X, y) = X^{-4}(\frac{1}{2}y^2 - \frac{1}{32}X^2 + \frac{1}{48}X)$,
 $M(X) = X^{-5}$.

(2) Case $-$: $X = 1 + 12x$, $H(X, y) = X^{-2/3}(\frac{1}{2}y^2 + \frac{1}{48}X + \frac{1}{96})$,
 $M(X) = X^{-5/3}$.

(vii) *The reversible Lotka-Volterra system* $Q_3^{LV} [\dot{z} = -iz + z^2 + b\bar{z}^2]$

(1) Case $b = -1$: $X = 1 + 4x$, $H(X, y) = \frac{1}{2}y^2 + \frac{1}{16}(X - \ln X)$, $M(X) = X^{-1}$.

(2) Case $b = 0$: $X = 1 + 2x$, $H(X, y) = X^{-1} [\frac{1}{2}y^2 + \frac{1}{8}(X + 1)^2]$, $M(X) = X^{-2}$.

(3) Case $b = \frac{1}{3}$: $X = 1 + \frac{4}{3}x$, $H(X, y) = X^{-2} [\frac{1}{2}y^2 + \frac{9}{16}(X^2 \ln X + X)]$, $M(X) = X^{-3}$.

(4) Case $b = 1$: $H(x, y) = e^{-4x} [\frac{1}{2}y^2 - \frac{1}{2}(x + \frac{1}{2})^2]$, $M(x) = e^{-4x}$.

(5) The general case: $X = 1 - 2(b - 1)x$,

$$H(X, y) = X^{\frac{b+1}{b-1}} \left(\frac{y^2}{2} - \frac{b+1}{8(3b-1)(b-1)^2} \left(X - \frac{3b-1}{b+1} \right)^2 \right),$$

$$M(X) = X^{\frac{2}{b-1}}.$$

(viii) *The Hamiltonian triangle* $[\dot{z} = -iz + \bar{z}^2]$: $X = 1 - 2x$,
 $H(X, y) = X [\frac{1}{2}y^2 - \frac{1}{24}(X - 3)^2]$.

REFERENCES

- [1] BAUTIN (N. N.). – On the number of limit cycles which appear with the variations of the coefficients from an equilibrium point of focus or center type, in: *AMS Translations-Series 1*, 5, 1962, p. 396-413 [Russian original: *Math. Sbornik*, 30, 1952, p. 181-196].
- [2] CHICONE (C.). – On bifurcation of limit cycles from centers, in: *Bifurcations of planar vector fields*, J.-P. FRANÇOISE et R. ROUSSARIE (eds.). – Berlin, Springer, 1990 (*Lect. Notes in Mathematics*, 1455, p. 20-43).
- [3] CHICONE (C.) and JACOBS (M.). – Bifurcations of limit cycles from quadratic isochrones, *J. Differential Equations*, 91, 1991, p. 268-326.
- [4] COPPEL (W. A.). – Some quadratic systems with at most one limit cycle, *Dynamics Reported*, 2, 1989, p. 61-88.
- [5] FRANÇOISE (J.-P.). – Successive derivatives of a first return map, application to the study of quadratic vector fields (Preprint, October 1993), *Ergod. Theory & Dyn. Syst.*, 16, 1996, p. 87-96.
- [6] GAVRILOV (L.) and HOROZOV (E.). – Limit cycles and zeroes of Abelian integrals satisfying third order Picard-Fuchs equations, in *Bifurcations of planar vector fields*, J.-P. FRANÇOISE and R. ROUSSARIE (eds.). – Berlin, Springer, 1990 (*Lect. Notes in Mathematics*, 1455, p. 160-186).
- [7] HOROZOV (E.) and ILIEV (I. D.). – On the number of limit cycles in perturbations of quadratic Hamiltonian systems, *Proc. Lond. Math. Soc.*, 69, 1994, p. 198-224.

- [8] HOROZOV (E.) and ILIEV (I. D.). – Perturbations of quadratic Hamiltonian systems with symmetry, *Ann. Inst. Henri Poincaré, Analyse non linéaire*, 13, 1996, p. 17-56.
- [9] ILIEV (I. D.). – Higher order Melnikov functions for degenerate cubic Hamiltonians (preprint 1994), *Adv. Differential Equations*, 1, 1996, p. 689-708.
- [10] ILIEV (I. D.). – The cyclicity of the period annulus of the quadratic Hamiltonian triangle, *J. Differential equations*, 128, 1996, p. 309-326.
- [11] IL'YASHENKO (Yu. S.). – Number of zeros of certain Abelian integrals in a real domain, *Funct. Anal. Appl.*, 11, 1978, p. 78-79.
- [12] KAPTEIN (W.). – New investigations on the midpoints of integrals of differential equations of the first order and the first degree, *Koninkl. Nederl. Wetensch. Verslagen*, 20, 1912, p. 1354-1465, 21, p. 27-33 (in Dutch).
- [13] LI (C.) and ROUSSEAU (C.). – A system with three limit cycles appearing in a Hopf bifurcation and dying in a homoclinic bifurcation: The cusp of order 4, *J. Differential Equations*, 79, 1989, p. 132-167.
- [14] LOUD (W. S.). – Behaviour of the periods of solutions of certain plane autonomous systems near centers, *Contrib. Differential Equations*, 3, 1964, p. 21-36.
- [15] MARDEŠIĆ (P.). – The number of limit cycles of polynomial perturbations of a Hamiltonian vector field, *Ergod. Th. & Dynam. Sys.*, 10, 1990, p. 523-529.
- [16] PETROV (G. S.). – On the number of zeros of complete elliptic integrals, *Funct. Anal. Appl.*, 18, 1984, p. 73-74.
- [17] PETROV (G. S.). – Elliptic integrals and their non-oscillatormess, *Funct. Anal. Appl.*, 20, 1986, p. 46-49.
- [18] ROUSSEAU (C.). – Universal unfolding of a singularity of a symmetric vector field with 7-jet C^∞ -equivalent to $y \partial/\partial x + (\pm x^3 \pm x^6 y) \partial/\partial y$, in *Bifurcations of planar vector fields*, J.-P. FRANÇOISE and R. ROUSSARIE (eds.). – Berlin, Springer, 1990 (*Lect. Notes in Mathematics*, 1455, p. 334-355).
- [19] ROUSSEAU (C.) and ŻOŁĄDEK (H.). – Zeros of complete elliptic integrals for 1:2 resonance, *J. Differential Equations*, 94, 1991, p. 41-54.
- [20] SHAFER (D. S.) and ZEGELING (A.). – Bifurcation of limit cycles from quadratic centers, *J. Differential Equations*, 122, 1995, p. 48-70.
- [21] SIBIRSKII (K. S.). – *Introduction to the algebraic theory of invariants of differential equations*. – Manchester Univ. Press, 1988 [Russian Original: Stiintsa, Kishinev, 1976].
- [22] VAN GILS (S. A.). – An inhomogeneous Picard-Fuchs equation, in *Dynamics, Bifurcation and Symmetry*, P. CHOSSAT ed., p. 333-341. – Kluwer, 1994.
- [23] VAN HORSSSEN (W. T.) and KOOLJ (R. E.). – Bifurcations of limit cycles in a particular class of quadratic systems with two centres, *J. Differential Equations*, 114, 1994, p. 538-569.
- [24] YAKOVENKO (S.). – A geometric proof of the Bautin theorem, in: Concerning the Hilbert 16th problem, *AMS Translations-Series*, 2, 165, 1995.
- [25] YUE HE and CHENGZHI LI. – *On the number of limit cycles arising from perturbation of homoclinic loops of quadratic integrable systems* (Preprint Peking Univ., 1995).
- [26] ZHI-FEN ZHANG and BAO-YI LI. – A note on a result of G. S. Petrov about the weakened 16th Hilbert problem, *J. Math. Anal. Appl.*, 190, 1995, p. 489-516.
- [27] ZHI-FEN ZHANG and CHENGZHI LI. – On the number of limit cycles of a class of quadratic Hamiltonian systems under quadratic perturbations, *Res. Rept. Peking Univ.*, No. 33, 1993.

- [28] ŻOŁĄDEK (H.). – Quadratic systems with center and their perturbations, *J. Differential Equations*, 109, 1994, p. 223-273.
- [29] ŻOŁĄDEK (H.). – Eleven small limit cycles in a cubic vector field, *Nonlinearity*, 8, 1995, p. 843-860.
- [30] ŻOŁĄDEK (H.). – The cyclicity of triangles and segments in quadratic systems, *J. Differential Equations*, 122, 1995, p. 137-159.
- [31] ŻOŁĄDEK (H.). – *Asymptotic properties of Abelian integrals arising in quadratic systems*, (Preprint, Univ. of Warsaw, 1994).